Graded Homework 1 (due Monday, March 19, 1:15 PM)

The first two exercises show that classical matrix inequalities may be recovered from known facts regarding differential entropy.

Exercise 1. Show Hadamard's inequality: if A is an $n \times n$ positive semi-definite matrix, then

$$\det(A) \leq \prod_{j=1}^{n} a_{jj}$$

Hint: Use the following fact for the proof: if $X_1, \ldots, X_n$ are jointly continuous and complex-valued random variables, then

$$h(X_1, \ldots, X_n) \leq \sum_{j=1}^{n} h(X_j)$$

Exercise 2. Show that the map $A \mapsto \log\det(A)$ is concave on the set of $n \times n$ positive definite matrices.

Hint: Consider two n-variate complex-valued random vectors X and Y with different covariance matrices, and consider Z such that $Z = X$ with probability p, $Z = Y$ with probability $1 - p$. Use then the two following facts: a) conditioning reduces entropy, b) the differential entropy of a random vector with a given covariance matrix is maximized when the vector is Gaussian.

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Let us now consider a multiple antenna channel with random fading matrix H varying ergodically over time, known at the receiver, but not at the transmitter. Let us define the function

$$\psi(Q) = \mathbb{E}_H(\log\det(I + HQH^*))$$

over the set of $n \times n$ positive semi-definite matrices Q. We say that Q_{opt} is optimal if

$$\psi(Q_{opt}) = \sup_{Q \geq 0 \,:\, \mathrm{Tr}(Q) \leq P} \psi(Q)$$

The following two exercises show that if the distribution of H exhibits some symmetry, then something can be said on the shape of Q_{opt}.

Exercise 3. a) Show that if the channel coefficients h_{jk} are i.i.d., then $\psi(\Pi Q \Pi^*) = \psi(Q)$, for any $Q \geq 0$ and any permutation matrix Π (whose entries are given by $\pi_{jk} = \delta_{j,\sigma(k)}$ for a given permutation σ on {1,. . . ,n}).

b) Deduce from a) and Exercise 2 that in this case, Q_{opt} is of the form

$$(Q_{opt})_{jk} = \begin{cases} P/n, & \text{if } j = k \\ Pc/n, & \text{if } j \neq k \end{cases}$$

where $-1/(n-1) \leq c \leq 1$ is some parameter (show that this last condition on c guarantees that $Q_{opt} \geq 0$).

Exercise 4. a) Show that if the channel coefficients h_{jk} are independent and such that for all j, k, $-h_{j,k}$ has the same distribution as h_{jk}, then $\psi(\Sigma Q \Sigma^*) = \psi(Q)$, for any $Q \geq 0$ and any matrix $\Sigma = \mathrm{diag}(\pm 1, \ldots, \pm 1)$.

b) Deduce from a) and Exercise 2 that in this case, Q_{opt} is diagonal.