The first two exercises show that classical matrix inequalities may be recovered from known facts regarding differential entropy.

**Exercise 1.** Show Hadamard’s inequality: if \( A \) is an \( n \times n \) positive semi-definite matrix, then
\[
\det(A) \leq \prod_{j=1}^{n} a_{jj}
\]
*Hint:* Use the following fact for the proof: if \( X_1, \ldots, X_n \) are jointly continuous and complex-valued random variables, then
\[
h(X_1, \ldots, X_n) \leq \sum_{j=1}^{n} h(X_j)
\]

**Exercise 2.** Show that the map \( A \mapsto \log \det(A) \) is concave on the set of \( n \times n \) positive definite matrices.

*Hint:* Consider two \( n \)-variate complex-valued random vectors \( X \) and \( Y \) with different covariance matrices, and consider \( Z \) such that \( Z = X \) with probability \( p \), \( Z = Y \) with probability \( 1 - p \). Use then the two following facts: a) conditioning reduces entropy; b) the differential entropy of a random vector with a given covariance matrix is maximized when the vector is Gaussian.

Let us now consider a multiple antenna channel with random fading matrix \( H \) varying ergodically over time, known at the receiver, but not at the transmitter. Let us define the function
\[
\psi(Q) = \mathbb{E}_H(\log \det(I + HQH^*))
\]
over the set of \( n \times n \) positive semi-definite matrices \( Q \). We say that \( Q_{opt} \) is optimal if
\[
\psi(Q_{opt}) = \sup_{Q \geq 0 : \tr(Q) \leq P} \psi(Q)
\]
The following two exercises show that if the distribution of \( H \) exhibits some symmetry, then something can be said on the shape of \( Q_{opt} \).

**Exercise 3.** a) Show that if the channel coefficients \( h_{jk} \) are i.i.d., then \( \psi(\Pi Q \Pi^*) = \psi(Q) \), for any \( Q \geq 0 \) and any permutation matrix \( \Pi \) (whose entries are given by \( \pi_{jk} = \delta_{j,\sigma(k)} \) for a given permutation \( \sigma \) on \( \{1, \ldots, n\} \)).

b) Deduce from a) and Exercise 2 that in this case, \( Q_{opt} \) is of the form
\[
(Q_{opt})_{jk} = \begin{cases} P/n, & \text{if } j = k \\ Pc/n, & \text{if } j \neq k \end{cases}
\]
where \(-1/(n-1) \leq c \leq 1\) is some parameter (show that this last condition on \( c \) guarantees that \( Q_{opt} \geq 0 \)).

**Exercise 4.** a) Show that if the channel coefficients \( h_{jk} \) are independent and such that for all \( j, k \), \(-h_{j,k}\) has the same distribution as \( h_{jk} \), then \( \psi(SQ\Sigma^*) = \psi(Q) \), for any \( Q \geq 0 \) and any matrix \( \Sigma = \text{diag}(\pm 1, \ldots, \pm 1) \).

b) Deduce from a) and Exercise 2 that in this case, \( Q_{opt} \) is diagonal.