The first two exercises show that classical matrix inequalities may be recovered from known facts regarding differential entropy.

**Exercise 1.** Show Hadamard’s inequality: if $A$ is an $n \times n$ positive semi-definite matrix, then
\[
\det(A) \leq \prod_{j=1}^{n} a_{jj}
\]

*Hint:* Use the following fact for the proof: if $X_1, \ldots, X_n$ are jointly continuous and complex-valued random variables, then
\[
h(X_1, \ldots, X_n) \leq \sum_{j=1}^{n} h(X_j)
\]

**Exercise 2.** Show that the map $A \mapsto \log \det(A)$ is concave on the set of $n \times n$ positive definite matrices.

*Hint:* Consider two $n$-variate complex-valued random vectors $X$ and $Y$ with different covariance matrices, and consider $Z$ such that $Z = X$ with probability $p$, $Z = Y$ with probability $1 - p$. Use then the two following facts: a) conditioning reduces entropy; b) the differential entropy of a random vector with a given covariance matrix is maximized when the vector is Gaussian.

Let us now consider a multiple antenna channel with random fading matrix $H$ varying ergodically over time, known at the receiver, but not at the transmitter. Let us define the function
\[
\psi(Q) = \mathbb{E}_{H}(\log \det(I + HQH^*))
\]
over the set of $n \times n$ positive semi-definite matrices $Q$. We say that $Q_{\text{opt}}$ is optimal if
\[
\psi(Q_{\text{opt}}) = \sup_{Q \geq 0 : \text{Tr}(Q) \leq P} \psi(Q)
\]
The following two exercises show that if the distribution of $H$ exhibits some symmetry, then something can be said on the shape of $Q_{\text{opt}}$.

**Exercise 3.** a) Show that if the channel coefficients $h_{jk}$ are i.i.d., then $\psi(\Pi Q \Pi^*) = \psi(Q)$, for any $Q \geq 0$ and any permutation matrix $\Pi$ (whose entries are given by $\pi_{jk} = \delta_{j,\sigma(k)}$ for a given permutation $\sigma$ on $\{1, \ldots, n\}$).

b) Deduce from a) and Exercise 2 that in this case, $Q_{\text{opt}}$ is of the form
\[
(Q_{\text{opt}})_{jk} = \begin{cases} P/n, & \text{if } j = k \\ \rho c/n, & \text{if } j \neq k \end{cases}
\]
where $-1/(n-1) \leq c \leq 1$ is some parameter (show that this last condition on $c$ guarantees that $Q_{\text{opt}} \geq 0$).

**Exercise 4.** a) Show that if the channel coefficients $h_{jk}$ are independent and such that for all $j, k$, $-h_{j,k}$ has the same distribution as $h_{j,k}$, then $\psi(\Sigma Q \Sigma^*) = \psi(Q)$, for any $Q \geq 0$ and any matrix $\Sigma = \text{diag}(\pm 1, \ldots, \pm 1)$.

b) Deduce from a) and Exercise 2 that in this case, $Q_{\text{opt}}$ is diagonal.