Homework November 25, 2011. Quantum information theory and computation

Problem 1. Example of comparison between Von Neumann and Shannon entropies

Suppose \( \rho = p |0\rangle \langle 0| + \frac{1-p}{2} (|0\rangle + |1\rangle)(\langle 0| + \langle 1|). \) Evaluate \( S(\rho) \) and compare with the classical Shannon entropy corresponding to \( \{p, 1-p\} \).

Problem 2. Von Neumann Entropy of a tensor product

Consider a composite system with tensor product density matrix \( \rho \otimes \sigma \). Prove that

\[ S(\rho \otimes \sigma) = S(\rho) + S(\sigma) \]

Problem 3. Entanglement and negative conditional entropy

Consider a pure state \( |AB\rangle \) of a composite system (say shared by Alice and Bob). Prove that \( |AB\rangle \) is entangled if and only if the conditional Von Neumann entropy is strictly negative.

Problem 4. Analog of Araki-Lieb inequality for conditional entropy

First show that for three random variables with any joint distribution the Shannon entropy always satisfies

\[ H(X,Y|Z) \geq H(X|Z) \]

Consider now a tripartite quantum system with Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \). Show that for quantum entropies it is not always true that \( S(A,B|C) \geq S(A|C) \). Prove that instead the following is always true

\[ S(A,B|C) \geq |S(A|C) - S(B|C)| \]