**Exercises November 11, 2011. Quantum information theory and computation**

**Problem 1. Mixtures**

a) Show that the two mixtures $\{|0\rangle, \frac{1}{2}; |1\rangle, \frac{1}{2}\}$ and $\{\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle), \frac{1}{2}; \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle), \frac{1}{2}\}$ have the same density matrix.

b) Consider the mixture $\{|0\rangle, \frac{1}{2}; \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{2}\}$. Give the spectral decomposition of the density matrix.

**Problem 2. Reduced density matrix**

a) Take the first GHZ state for three Qbits $\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Compute the reduced density matrices $\rho_{AB}$ and $\rho_C$.

b) Take the Bell state $|\Phi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Compute Alice's and Bob's reduced density matrices.

c) Check that the Schmidt decomposition theorem holds in each of the above cases.

**Problem 3. Schmidt decomposition theorem**

Consider the pure $N$ Qbit state,

$$|\Psi\rangle = \frac{1}{2^{N/2}} \sum_{b_1...b_N \in \{0,1\}^N} |b_1...b_N\rangle$$

a) Compute the density matrix of the first Qbit. Show that it has non degenerate eigenvalues 0 and 1.

b) Compute the reduced density matrix of the set of bits $(2...N)$. Show that this $2^{N-1} \times 2^{N-1}$ matrix has a non degenerate eigenvalue 1 and an eigenvalue 0 with degeneracy $2^{N-1} - 1$.

c) Check explicitly that the Schmidt decomposition theorem holds.

**Problem 4. Remarks about purification**

Consider a truly mixed state: one that is not extremal in the convex set of density matrices. Is it possible to purify with a pure tensor product state ? Is the purification unique ?