Exercises November 11, 2011. Quantum information theory and computation

Problem 1. Mixtures

a) Show that the two mixtures \(|0\rangle; \frac{1}{2}, |1\rangle; \frac{1}{2}\) and \(|\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{2}, \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \frac{1}{2}\) have the same density matrix.

b) Consider the mixture \(|0\rangle; \frac{1}{2}, \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{2}\}. Give the spectral decomposition of the density matrix.

Problem 2. Reduced density matrix

a) Take the first GHZ state for three Qbits \(|000\rangle + |111\rangle\) in the Hilbert space \(H_A \otimes H_B \otimes H_C\). Compute the reduced density matrices \(\rho_{AB}\) and \(\rho_C\).

b) Take the Bell state \(|\psi\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle\). Compute Alice's and Bob's reduced density matrices.

c) Check that the Schmidt decomposition theorem holds in each of the above cases.

Problem 3. Schmidt decomposition theorem

Consider the pure \(N\) Qbit state,

\[|\psi\rangle = \frac{1}{2^{N/2}} \sum_{b_1...b_N \in \{0,1\}^N} |b_1...b_N\rangle\]

a) Compute the density matrix of the first Qbit. Show that it has non degenerate eigenvalues 0 and 1.

b) Compute the reduced density matrix of the set of bits (2...N). Show that this \(2^{N-1} \times 2^{N-1}\) matrix has a non degenerate eigenvalue 1 and an eigenvalue 0 with degeneracy \(2^{N-1} - 1\).

c) Check explicitly that the Schmidt decomposition theorem holds.

Problem 4. Remarks about purification

Consider a truly mixed state; one that is not extremal in the convex set of density matrices. Is it possible to purify with a pure tensor product state? Is the purification unique?