ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Principles of Digital Communications:	Assignment date: Apr 20, 2011
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Solution of Homework 9

Problem 1. It's about the project.

Problem 2. *(Sampling And Reconstruction)*
For the zero-order interpolator we have:

1. It follows from the following chain of equalities where we use the facts that $p_0(t) \star \delta(t - nT) = p_0(t - nT)$ and the linearity of the convolution. The latter implies that the convolution may be brought in and out of a sum:

$$\begin{aligned} x_0(t) &= \sum x(nT)p_0(t - nT) \\ &= \sum x(nT)[\delta_t - nT) \star p_0(t)] \\ &= [\sum x(nT)\delta(t - nT)] \star p_0(t) \\ &= [x(t)E_T(t)] \star p_0(t). \end{aligned}$$

2. The Fourier transform of $x_0(t) = [x(t)E_T(t)] \star p_0(t)$ is

$$\begin{aligned} x_{0\mathcal{F}}(f) &= [x_{\mathcal{F}}(f) \star \frac{1}{T}E_{\frac{1}{T}}(f)]p_{0\mathcal{F}}(f) \\ &= [\sum x_{\mathcal{F}}(f - \frac{n}{T})]p_{0\mathcal{F}}(f). \end{aligned}$$

   In the proof of the sampling theorem, i.e., when sampling is done using $E_T(t)$, we have a similar expression but without the multiplicative term $p_{0\mathcal{F}}(f)$. In that case we can use an ideal lowpass filter to remove all but the baseband term of $\sum x_{\mathcal{F}}(f - \frac{n}{T})$, obtaining (in the frequency domain) $x_{\mathcal{F}}(f)$. The undesirable effect of the multiplicative term $p_{0\mathcal{F}}(f)$ is to shape all the replicas of $x_{\mathcal{F}}(f)$ in $\sum x_{\mathcal{F}}(f - \frac{n}{T})$, including the baseband term. Hence there is some distortion that can not be removed by an ideal lowpass filter. The desirable side effect is that it reduces the amplitude of the other replicas of $x_{\mathcal{F}}(f)$. This makes it easier to essentially remove them via a non-ideal lowpass filter.

3. An accurate reconstruction becomes easier as B and T become smaller. We can see this in the time domain and in the frequency domain. B small implies slow variation in the time domain an in turns it implies that the stepwise reconstruction $x_0(t)$ closely tracks $x(t)$. The lowpass filter will remove the jumps and the result is a fairly accurate reproduction of the original. The analysis in the frequency domain is more insightful since it allows to separately account for the effect of B, T, and of the lowpass filter, namely: If B is small, the effect of the multiplicative term to $x_{\mathcal{F}}(f)$ in $\sum x_{\mathcal{F}}(f - \frac{n}{T})$ is small. In fact its absolute value is $|T\mathrm{sinc}(fT)|$ which is essentially flat around $f = 0$. If T is small, then the replicas of $x_{\mathcal{F}}(f)$ are further apart, which makes it easier to (essentially) remove them with a non-ideal lowpass filter.

Arguing similarly for the first-order interpolator we have:

1. This is immediately obvious from a picture. In fact for the interval $t \in [0, T]$, the sum of the two pulses $ap_1(t) + bp_1(t - T)$ is exactly the straight line that goes form a at $t = 0$ to b at $t = T$.

2. The analysis in the frequency domain follows exactly the same lines as for the zeroth-order case. The only difference is that $p_{0\mathcal{F}}(f)$ (which is a sinc) becomes $p_{1\mathcal{F}}(f)$ which is a sinc squared. The effect on $x_{\mathcal{F}}(f)$ becomes worse since the sinc squared decays faster as we move away from $f = 0$. However, this effect is negligible in both cases if B is small. On the other hand, the fact that the sinc squared decays faster is good news for the lowpass filter which has to work less hard to remove the replicas of $x_{\mathcal{F}}(f)$.

Problem 3. *(DC-to-DC Converter)*

1. Using Picket-Fence formula we have

$$p_\tau(f) = \tau\mathrm{sinc}(f\tau) \times \frac{1}{T}\sum_{-\infty}^{\infty} \delta(f - \frac{n}{T}) = \frac{\tau}{T}\sum_{-\infty}^{\infty} \mathrm{sinc}(\frac{n\tau}{T})\delta(f - \frac{n}{T})$$

Using the fact that multiplication in the time domain corresponds to convolution in the frequency domain we have $y_{\mathcal{F}}(f) = x_{\mathcal{F}}(f) \star \frac{\tau}{T}\sum_{-\infty}^{\infty} \mathrm{sinc}(\frac{n\tau}{T})\delta(f - \frac{n}{T})$ which gives $y_{\mathcal{F}}(f) = \frac{\tau}{T}\sum_{-\infty}^{\infty} \mathrm{sinc}(\frac{n\tau}{T})x_{\mathcal{F}}(f - \frac{n}{T})$.

2. The effect of the ideal sampling in the frequency domain is creating periodic replicas of $x_{\mathcal{F}}(f)$ in the frequency domain with period $\frac{1}{T}$ and scaled by $\frac{1}{T}$. In this case we have approximately the same effect but different replicas are scaled by $\frac{\tau}{T}\mathrm{sinc}(\frac{n\tau}{T})$.

3. If τ goes to T then $\mathrm{sinc}(\frac{n\tau}{T})$ will be zero for all n except $n = 0$ which is 1. Hence, in the summation only the first term or $x_{\mathcal{F}}(f)$ remains which is also obvious from time domain representation because in the time domain $\tau = T$ results in $p_\tau(t) = 1$ and so the output is the input itself. As τ approaches 0 all of the coefficients $\frac{\tau}{T}\mathrm{sinc}(\frac{n\tau}{T})$ go to zero and the output will be zero which is also obvious from the time domain.

4. Assume that our input signal is a very low frequency signal and in the extreme case a constant signal with amplitude A then the Fourier transform of $x_{\mathcal{F}}(f)$ will be $A\delta(f)$ if we input this signal to DC-to-DC Converter circuit then the output signal Fourier transform will be $\sum_{-\infty}^{\infty} \frac{A\tau}{T}\text{sinc}(\frac{n\tau}{T})\delta(f-\frac{n}{T})$. If using a lowpass filter we extract DC component of the signal its amplitude is $\frac{A\tau}{T}$ and by changing τ we can control this DC component.

Problem 4. *(Communication Link Design)*

1. Total transmission bandwidth is 10 MHz and we can send 10 Msps (mega symbol per second). The required bit rate is 40 Mbps (mega bit per second) and so every symbol should carry 4 bits. Hence, we can use 16-QAM constellation for transmission.

2. The symbol error probability of 16-QAM is approximately $4Q(\frac{d}{\sqrt{2N_0}})$ which is also an upper bound for the bit error probability and d is the minimum distance between adjacent points in the constellation. Using the inequality $Q(x) < \frac{1}{2}\exp(-\frac{x^2}{2}), \quad x > 0$ we can use $2\exp(-\frac{d^2}{4N_0})$. Setting the probability of the error to 10^{-5} we obtain $d^2 = 2.051 \times 10^{-19}$. The mean symbol energy of 16-QAM is easily checked to be $\frac{5d^2}{2}$ and so $\mathcal{E}_s = 5.127 \times 10^{-19}$. Every symbol carries 4 bits and so $\mathcal{E}_b = \frac{\mathcal{E}_s}{4} = 1.282 \times 10^{-19}$. Bit rate of the link is 40 $Mbps$ and hence required power at the input of the receiver will be $P = R_b\mathcal{E}_b = 5.127 \times 10^{-12}$ W.

3. The distance between the transmitter and the receiver is 5 Km and the attenuation factor of the cable is 16 dB/Km and so the total attenuation is 80 dB and accounting this attenuation the required transmission power is 5.127×10^{-4} W.