Solution of Homework 8

Problem 1. (Average Energy of PAM)

1. The pdf of $S$ can be written as $f_S(s) = \sum_{i=-\frac{m}{2}+1}^{m} \delta(s - (2i - 1)a)$ while the pdf of $U$ is $f_U(u) = \frac{1}{2a} 1_{[-a,a]}(u)$. As $S$ and $U$ are independent the pdf of $V = S + U$ is the convolution of $f_S$ and $f_U$. From a sketch of $f_S$ and $f_U$ we immediately see that $f_V$ is uniform in $[-ma, ma]$.

2. $U$ and $V$ have symmetric distribution around zero so the mean value of both is zero. $E\{V^2\} = \int_{-ma}^{ma} v^2 f_V(v) dv = \int_{-ma}^{ma} v^2 \frac{dv}{2ma} = \frac{m^2a^2}{3}$. Hence, $\text{var}(V) = \frac{m^2a^2}{3}$. By symmetry, $\text{var}(U) = \frac{a^2}{3}$.

3. $U$ and $S$ are independent random variables and so $\text{var}(V) = \text{var}(S + U) = \text{var}(S) + \text{var}(U)$. Hence, $\text{var}(S) = \frac{(m^2-1)a^2}{3}$.

4. Actually we have derived the expression for the average energy of PAM given in the Example 4.4.57 where the distance between the adjacent points is $d = 2a$.

Problem 2. (Pulse Amplitude Modulated Signals)

1. From the previous problem we know that the mean energy of the PAM constellation with distance $d = 2a$ is equal to $\frac{(m^2-1)a^2}{3}$. Replacing $a$ by $\frac{d}{2}$ we have $\mathcal{E}_s = \frac{(m^2-1)d^2}{12}$.

2. The received signal is

$$y(t) = s_i(t) + N(t)$$

where $N(t)$ is a white Gaussian noise process.

The ML detector passes the received signal into a filter with impulse response $\phi(-t)$. Let $y$ be the output at time $t = 0$. The decision is $i$ if $i$ is the index that minimizes $||Y - s_i||^2$. 
3. The conditional probabilities of error are

\[
\Pr\left( e|s_i = \frac{+(m-1)d}{2} \right) = \Pr\left( e|s_i = \frac{-(m-1)d}{2} \right) = \Pr\left( Z > \frac{d}{2} \right) = Q\left( \frac{d}{\sqrt{2N_0}} \right)
\]

\[
\Pr\left( e|s_i \neq \pm\frac{(m-1)d}{2} \right) = \Pr\left( (Z < \frac{-d}{2}) \cup (Z > \frac{d}{2}) \right) = 2Q\left( \frac{d}{\sqrt{2N_0}} \right)
\]

hence

\[
\Pr(e) = \frac{2}{m} \Pr\left( e|s_i = \frac{+(m-1)d}{2} \right) + \frac{m-2}{m} \Pr\left( e|s_i \neq \pm\frac{(m-1)d}{2} \right)
\]

\[
= \frac{2m-1}{m} Q\left( \frac{d}{\sqrt{2N_0}} \right).
\]

4. Let \( m = 2^k \), then \( E_s = E_s(k) = \frac{d^2}{12}(4^k - 1) \) and

\[
\frac{E_s(k+1)}{E_s(k)} \approx 4
\]

Problem 3. (Root-Mean Square Bandwidth)

1. If we define inner product of two functions, which may be complex valued, by \( \langle f, g \rangle = \int_{-\infty}^{\infty} f^*(t)g(t)dt \) then we have \( |\langle f, g \rangle|^2 \leq \langle f, f \rangle < \langle g, g \rangle \) by Schwartz inequality. It can also be checked that \( \langle f, g \rangle = \langle g, f \rangle^* \). Using this definition \( \{ \int_{-\infty}^{\infty} [g_1(t)g_2(t) + g_1(t)g_2^*(t)]dt \} = \langle g_1, g_2 \rangle + \langle g_2, g_1 \rangle = \langle g_1, g_2 \rangle + \langle g_2, g_1 \rangle^* = 2\Re\langle g_1, g_2 \rangle \). Hence, \( \{ \int_{-\infty}^{\infty} [g_1(t)g_2(t) + g_1(t)g_2^*(t)]dt \} \leq 4\Re\langle g_1, g_2 \rangle \leq 4 < g_1, g_1 > < g_2, g_2 > \) and writing the expression for \( < g_1, g_1 > \) and \( < g_2, g_2 > \) we have

\[
\left| \left\{ \int_{-\infty}^{\infty} [g_1^*(t)g_2(t) + g_1(t)g_2^*(t)]dt \right\} \right|^2 \leq 4 \int_{-\infty}^{\infty} |g_1(t)|^2 dt \int_{-\infty}^{\infty} |g_2(t)|^2 dt.
\]

2. Expanding the expression and using the fact that \( t \) is a real number we have

\[
\left[ \int_{-\infty}^{\infty} t \frac{d}{dt} [g(t)g^*(t)] dt \right]^2 = \left[ \int_{-\infty}^{\infty} [(tg(t))(g'(t))^* + (tg(t))^*g'(t)] dt \right]^2
\]

Using the result in the previous part and setting \( g_1(t) = tg(t) \) and \( g_2(t) = g'(t) \) we have

\[
\left[ \int_{-\infty}^{\infty} t \frac{d}{dt} [g(t)g^*(t)] dt \right]^2 \leq 4 \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} \left| \frac{dg(t)}{dt} \right|^2 dt
\]

. 2
3. Integrating by part we have
\[
\int_{-\infty}^{\infty} t \frac{d}{dt} [g(t)g^*(t)] \, dt = t|g(t)|^2|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} |g(t)|^2 \, dt
\]
First component is zero by the problem statement and so remains the second component. Hence, replacing in the result of previous part we have
\[
\left[ \int_{-\infty}^{\infty} |g(t)|^2 \, dt \right]^2 \leq 4 \int_{-\infty}^{\infty} t^2 |g(t)|^2 \, dt \int_{-\infty}^{\infty} \left| \frac{dg(t)}{dt} \right|^2 \, dt
\]
4. From Parseval’s relation we have
\[
\int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |G(f)|^2 \, df
\]
Further more we know that the Fourier transform of \( \frac{dg(t)}{dt} \) is \( j2\pi f G(f) \) and applying the Praseval’s relation to \( \frac{dg(t)}{dt} \) we have
\[
\int_{-\infty}^{\infty} \left| \frac{dg(t)}{dt} \right|^2 \, dt = \int_{-\infty}^{\infty} 4\pi^2 f^2 |G(f)|^2 \, df
\]
replacing in the result of the previous part we have
\[
\int_{-\infty}^{\infty} |g(t)|^2 \, dt \int_{-\infty}^{\infty} |G(f)|^2 \, df \leq (4\pi)^2 \int_{-\infty}^{\infty} t^2 |g(t)|^2 \, dt \int_{-\infty}^{\infty} f^2 |G(f)|^2 \, df
\]
5. Simply, dividing the right part of the equality in the previous part by the left part and using the definition of \( T_{rms} \) and \( W_{rms} \) we obtain \( T_{rms} W_{rms} \geq \frac{1}{4\pi} \).
6. For the Gaussian pulse, it is easily checked that the shape of the pulse squared is similar to the Gaussian distribution with \( \sigma^2 = \frac{1}{4\pi} \) and which also needs some normalization factor. Putting altogether we have
\[
T_{rms}^2 = \left[ \frac{\int_{-\infty}^{\infty} t^2 \exp(-\pi t^2)^2 \, dt}{\int_{-\infty}^{\infty} \exp(-\pi t^2)^2 \, dt} \right]
\]
\[
= \frac{\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} t^2 \exp(-t^2/2\sigma^2) \, dt}{\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp(-t^2/2\sigma^2) \, dt}
\]
\[
= \sigma^2
\]
\[
= \frac{1}{4\pi}
\]
Using the fact that
\[
\exp(-\pi t^2) \leftrightarrow \exp(-\pi f^2)
\]
we have
\[
W_{rms}^2 = \frac{1}{4\pi}
\]
Thus for the Gaussian pulse, we have
\[
T_{rms} W_{rms} = \frac{1}{4\pi}
\]
Problem 4. (Orthogonal Signal Sets)

1. To find the minimum-energy signal set, we first compute the centroid of the signal set:

\[ a = \sum_{j=0}^{m-1} P_H(j) s_j(t) = \frac{1}{m} \sum_{j=0}^{m-1} \sqrt{E_s} \phi_j(t). \]

so

\[ s_j^*(t) = s_j(t) - a \]

\[ = \sqrt{E_s} \phi_j(t) - \frac{1}{m} \sum_{i=0}^{m-1} \sqrt{E_s} \phi_i(t) \]

\[ = \sqrt{E_s} \frac{m-1}{m} \phi_j(t) - \frac{1}{m} \sum_{i \neq j} \sqrt{E_s} \phi_i(t). \]

2. Notice that \( \sum_{j=0}^{m-1} s_j^*(t) = 0 \) by the definition of \( s_j^*(t) \), \( j = 0, 1, \ldots, m-1 \). Hence, the \( m \) signals \( \{s_0^*(t), \ldots, s_{m-1}^*(t)\} \) are linearly dependent. This means that their space has dimensionality less than \( m \). We show that any collection of \( m-1 \) or less is linearly independent. That would prove that the dimensionality of the space \( \{s_0^*(t), \ldots, s_{m-1}^*(t)\} \) is \( m-1 \). Without loss of generality we consider \( s_0^*(t), \ldots, s_{m-2}^*(t) \). Assume that \( \sum_{j=0}^{m-2} \alpha_j s_j^*(t) = 0 \). Using the definition of \( s_j^*(t) \) \( j = 0, 1, \ldots, m-1 \) we may write

\[ \sum_{j=0}^{m-2} (\alpha_j - \beta) s_j(t) - \beta s_{m-1}(t) = 0 \]

where \( \beta = \frac{1}{m} \sum_{j=0}^{m-1} \alpha_j \). But \( s_0(t), s_1(t), \ldots, s_{m-1}(t) \) is an orthogonal set and this implies \( \beta = 0 \) and \( \alpha_j = \beta = 0 \) \( j = 0, 1, \ldots, m-2 \). That means that \( \alpha_j = 0 \) \( j = 0, 1, \ldots, m-2 \). Hence, \( s_j^*(t) \) \( j = 0, 1, \ldots, m-2 \) are linearly independent. We have proved that the new set spans a space of dimension \( m-1 \).

3. It is easy to show that n-tuple corresponding to \( s_j^* \) is \( \sqrt{E_s} \frac{m-1}{m} \) at position \( j \) and \( \frac{\sqrt{E_s}}{m} \) at all other positions. Clearly \( ||s_j^*||^2 = (m-1) \frac{E_s}{m^2} + \frac{E_s}{m} (m-1)^2 = E_s (1 - \frac{1}{m}) \). This is independent of \( j \) so the average energy is also \( E_s (1 - \frac{1}{m}) \).

Problem 5. (m-ary Frequency shift Keying)

1. Orthogonality requires \( \int_0^T \cos(2\pi(f_c + i\Delta f) t) \cos(2\pi(f_c + j\Delta f) t) dt = 0 \) for every \( i \neq j \). Using the trigonometric identity \( \cos(\alpha) \cos(\beta) = \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta) \), an equivalent condition is \( \frac{1}{2} \int_0^T \left[ \cos(2\pi(i - j) \Delta f) t + \cos(2\pi(2f_c + (i + j) \Delta f) t) \right] dt = 0 \).

Integrating we obtain

\[ \frac{\sin(2\pi(i-j)\Delta f T)}{2\pi(i-j)\Delta f} + \frac{\sin(2\pi(2f_c + (i + j) \Delta f) T)}{2\pi(2f_c + (i + j) \Delta f)} = 0. \]

As \( f_c T \) is assumed to be an integer, the result can be simplified to

\[ \sin(2\pi(i-j)\Delta f T) + \sin(2\pi(i+j)\Delta f T) = 0. \]

As \( i \) and \( j \) are integer this result is zeros for \( i \neq j \) if and only if \( 2\pi \Delta f T \) is an integer multiple of \( \pi \). Hence, we obtain the minimum value of \( \Delta f \) if \( 2\pi \Delta f T = \pi \) which gives \( \Delta f = \frac{1}{2T} \).
2. Proceeding similarly we will have orthogonality if and only if
\[
\frac{\sin(2\pi(i-j)\Delta f T + \theta_i - \theta_j) - \sin(\theta_i - \theta_j)}{2\pi(i-j)\Delta f} = 0.
\]
In this case we see that both parts become zero if and only if \(2\pi \Delta f T\) is an even multiple of \(\pi\) which means that the smallest \(\Delta f\) is \(\Delta f = \frac{1}{T}\) which is twice the minimum frequency separation needed in the previous part. Hence, the cost of phase uncertainty is a bandwidth expansion by a factor of 2.

3. The condition we obtained for the orthogonality in the first part consist of two terms as follows
\[
\int_0^T [\cos(2\pi(i-j)\Delta f t) + \cos(2\pi(2f_c + (i+j)\Delta f) t)] dt = 0.
\]
We saw that if \(f_c T\) is exactly an integer number then with have orthogonality with \(\Delta f = \frac{1}{2T}\). Now assume that \(f_c >> M\Delta f\) in this case the integral value will be
\[
\frac{\sin(2\pi(2f_c + (i+j)\Delta f) T)}{2\pi(2f_c + (i+j)\Delta f)}
\]
which its absolute value is always less that \(\frac{1}{2\pi(2f_c + (i+j)\Delta f)}\) which approaches zero as \(f_c\) becomes bigger and bigger. So if we choose \(\Delta f = \frac{1}{2T}\) and take \(f_c >> m\Delta f\) then we will have approximately orthogonality. In a similar way when we have a random phase shift then we can choose \(\Delta f = \frac{1}{2}\) and take \(f_c >> m\Delta f\) to have orthogonality.

4. Integrating \(s_i(t)^2\) over \([0,T]\) we obtain \(A^2 \times \frac{2}{T} \times \frac{1}{2} \times T = A^2\) which holds for every \(i\). Hence, the mean energy of the constellation is \(A^2\) but this energy is transmitted during \([0,T]\) so the mean power will be \(\frac{A^2}{T}\) which is independent of \(k\).

5. We have \(M\) signals separated by \(\Delta f\). The approximate bandwidth is \(m\Delta f\). This means bandwidth \(\frac{2k}{T}\) in the former case, without random phase shift, and bandwidth \(\frac{4k}{T}\) in the latter case in which we have a random phase shift.

6. Practical systems have a constant \(B\) and a \(T\) which grows linearly with \(k\). Even if we let \(T\) grow linearly with \(k\), in the system considered here, \(B\) grows exponentially with \(k\). This is not practical.