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Solution of Homework 8

Problem 1. *(Average Energy of PAM)*

1. The pdf of S can be written as $f_S(s) = \sum_{i=-\frac{m}{2}+1}^{\frac{m}{2}} \delta(s-(2i-1)a)$ while the pdf of U is $f_U(u) = \frac{1}{2a}\mathbf{1}_{[-a,a]}(u)$. As S and U are independent the pdf of $V = S+U$ is the convolution of f_S and f_U. From a sketch of f_S and f_U we immediately see that f_V is uniform in $[-ma, ma]$.

2. U and V have symmetric distribution around zero so the mean value of both is zero. $E\{V^2\} = \int_{-ma}^{ma} v^2 f_V(v)dv = \int_{ma}^{ma} v^2 \frac{dv}{2ma} = \frac{m^2a^2}{3}$. Hence, $\text{var}(V) = \frac{m^2a^2}{3}$. By symmetry, $\text{var}(U) = \frac{a^2}{3}$.

3. U and S are independent random variables and so $\text{var}(V) = \text{var}(S+U) = \text{var}(S) + \text{var}(U)$. Hence, $\text{var}(S) = \frac{(m^2-1)a^2}{3}$.

4. Actually we have derived the expression for the average energy of PAM given in the Example 4.4.57 where the distance between the adjacent points is $d = 2a$.

Problem 2. *(Pulse Amplitude Modulated Signals)*

1. From the previous problem we know that the mean energy of the PAM constellation with distance $d = 2a$ is equal to $\frac{(m^2-1)a^2}{3}$. Replacing a by $\frac{d}{2}$ we have $\mathcal{E}_s = \frac{(m^2-1)d^2}{12}$.

2. The received signal is

$$y(t) = s_i(t) + N(t)$$

where $N(t)$ is a white Gaussian noise process.

The ML detector passes the received signal into a filter with impulse response $\phi(-t)$. Let y be the output at time $t = 0$. The decision is i if i is the index that minimizes $||Y - s_i||^2$.

3. The conditional probabilities of error are

$$\begin{aligned}
\Pr\left(e|s_i = \frac{+(m-1)d}{2}\right) &= \Pr\left(e|s_i = \frac{-(m-1)d}{2}\right) \\
&= \Pr\left(Z > \frac{d}{2}\right) = Q\left(\frac{d}{\sqrt{2N_0}}\right) \\
\Pr\left(e|s_i \neq \frac{\pm(m-1)d}{2}\right) &= \Pr\left((Z < \frac{-d}{2}) \cup (Z > \frac{d}{2})\right) = 2Q\left(\frac{d}{\sqrt{2N_0}}\right)
\end{aligned}$$

hence

$$\begin{aligned}
\Pr(e) &= \frac{2}{m}\Pr\left(e|s_i = \frac{+(m-1)d}{2}\right) + \frac{m-2}{m}\Pr\left(e|s_i \neq \frac{\pm(m-1)d}{2}\right) \\
&= 2\frac{m-1}{m}Q\left(\frac{d}{\sqrt{2N_0}}\right).
\end{aligned}$$

4. Let $m = 2^k$, then $\mathcal{E}_s = \mathcal{E}_s(k) = \frac{d^2}{12}(4^k - 1)$ and

$$\frac{E_s(k+1)}{E_s(k)} \simeq 4$$

Problem 3. *(Root-Mean Square Bandwidth)*

1. If we define inner product of two function, which may be complex valued, by $< f, g > \triangleq \int_{-\infty}^{\infty} f^\star(t)g(t)dt$ then we have $| < f, g > |^2 \leq < f, f >< g, g >$ by Schwartz inequality. It can also be checked that $< f, g >=< g, f >^\star$. Using this definition $\left\{\int_{-\infty}^{\infty}[g_1^*(t)g_2(t) + g_1(t)g_2^*(t)]dt\right\} =< g_1, g_2 > + < g_2, g_1 >=< g_1, g_2 > + < g_1, g_2 >^\star= 2\Re(< g_1, g_2 >)$. Hence, $|\left\{\int_{-\infty}^{\infty}[g_1^*(t)g_2(t) + g_1(t)g_2^*(t)]dt\right\}|^2 = 4|\Re(< g_1, g_2 >)|^2 \leq 4 < g_1, g_1 >< g_2, g_2 >$ and writing the expression for $< g_1, g_1 >$ and $< g_2, g_2 >$ we have $|\left\{\int_{-\infty}^{\infty}[g_1^*(t)g_2(t) + g_1(t)g_2^*(t)]dt\right\}|^2 \leq 4\int_{-\infty}^{\infty}|g_1(t)|^2dt\int_{-\infty}^{\infty}|g_2(t)|^2dt$.

2. Expanding the expression and using the fact that t is a real number we have

$$\left[\int_{-\infty}^{\infty} t\frac{d}{dt}\left[g(t)g^*(t)\right]dt\right]^2 = \left[\int_{-\infty}^{\infty}[(tg(t))(g^{'}(t))^\star + (tg(t))^\star g^{'}(t)]dt\right]^2$$

Using the result in the previous part and setting $g_1(t) = tg(t)$ and $g_2(t) = g^{'}(t)$ we have

$$\left[\int_{-\infty}^{\infty} t\frac{d}{dt}\left[g(t)g^*(t)\right]dt\right]^2 \leq 4\int_{-\infty}^{\infty} t^2|g(t)|^2dt \int_{-\infty}^{\infty}\left|\frac{dg(t)}{dt}\right|^2 dt$$

.

3. Integrating by part we have

$$\int_{-\infty}^{\infty} t\frac{d}{dt}\left[g(t)g^*(t)\right]dt = t|g(t)|^2|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}|g(t)|^2dt$$

First component is zero by the problem statement and so remains the second component. Hence, replacing in the result of previous part we have

$$\left[\int_{-\infty}^{\infty}|g(t)|^2dt\right]^2 \leq 4\int_{-\infty}^{\infty}t^2|g(t)|^2dt\int_{-\infty}^{\infty}\left|\frac{dg(t)}{dt}\right|^2dt$$

4. From Parseval's relation we have

$$\int_{-\infty}^{\infty}|g(t)|^2dt = \int_{-\infty}^{\infty}|G(f)|^2df$$

Further more we know that the Fourier transform of $\frac{dg(t)}{dt}$ is $j2\pi fG(f)$ and applying the Praseval's relation to $\frac{dg(t)}{dt}$ we have

$$\int_{-\infty}^{\infty}|\frac{dg(t)}{dt}|^2dt = \int_{-\infty}^{\infty}4\pi^2f^2|G(f)|^2df$$

replacing in the result of the previous part we have

$$\int_{-\infty}^{\infty}|g(t)|^2dt\int_{-\infty}^{\infty}|G(f)|^2df \leq (4\pi)^2\int_{-\infty}^{\infty}t^2|g(t)|^2dt\int_{-\infty}^{\infty}f^2|G(f)|^2df$$

5. Simply, dividing the right part of the equality in the previous part by the left part and using the definition of T_{rms} and W_{rms} we obtain $T_{rms}W_{rms} \geq \frac{1}{4\pi}$.

6. For the Gaussian pulse, it is easily checked that the shape of the pulse squared is similar to the Gaussian distribution with $\sigma^2 = \frac{1}{4\pi}$ and which also needs some normalization factor. Putting altogether we have

$$\begin{aligned}
T_{rms}^2 &= \left[\frac{\int_{-\infty}^{\infty}t^2|\exp(-\pi t^2)|^2dt}{\int_{-\infty}^{\infty}|\exp(-\pi t^2)|^2dt}\right] \\
&= \left[\frac{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}t^2\exp(-t^2/2\sigma^2)dt}{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp(-t^2/2\sigma^2)dt}\right] \\
&= \sigma^2 \\
&= \frac{1}{4\pi}
\end{aligned}$$

Using the fact that

$$\exp(-\pi t^2) \xleftrightarrow{\mathcal{F}} \exp(-\pi f^2).$$

we have

$$W_{rms}^2 = \frac{1}{4\pi}.$$

Thus for the Gaussian pulse, we have

$$T_{rms}W_{rms} = \frac{1}{4\pi}.$$

Problem 4. *(Orthogonal Signal Sets)*

1. To find the minimum-energy signal set, we first compute the centroid of the signal set:

$$a = \sum_{j=0}^{m-1} P_H(j) s_j(t) = \frac{1}{m} \sum_{j=0}^{m-1} \sqrt{\mathcal{E}_s}\phi_j(t).$$

so

$$\begin{aligned} s_j^*(t) &= s_j(t) - a \\ &= \sqrt{\mathcal{E}_s}\phi_j(t) - \frac{1}{m}\sum_{i=0}^{m-1}\sqrt{\mathcal{E}_s}\phi_i(t) \\ &= \sqrt{\mathcal{E}_s}\frac{m-1}{m}\phi_j(t) - \frac{1}{m}\sum_{i \neq j}\sqrt{\mathcal{E}_s}\phi_i(t). \end{aligned}$$

2. Notice that $\sum_{j=0}^{m-1} s_j^\star(t) = 0$ by the definition of $s_j^\star(t), \quad j = 0, 1, \cdots, m-1$. Hence, the m signals $\{s_0^\star(t), \cdots, s_{m-1}^\star(t)\}$ are linearly dependent. This means that their space has dimensionality less than m. We show that any collection of $m-1$ or less is linearly independent. That would prove that the dimensionality of the space $\{s_0^\star(t), \cdots, s_{m-1}^\star(t)\}$ is $m-1$. Without loss of generality we consider $s_0^\star(t), \cdots, s_{m-2}^\star(t)$. Assume that $\sum_{j=0}^{m-2} \alpha_j s_j^\star(t) = 0$. Using the definition of $s_j^\star(t) \quad j = 0, 1, \cdots, m-1$ we may write $\sum_{j=0}^{m-2}(\alpha_j - \beta)s_j(t) - \beta s_{m-1}(t) = 0$ where $\beta = \frac{1}{m}\sum_{j=0}^{m-1}\alpha_j$. But $s_0(t), s_1(t), \cdots, s_{m-1}(t)$ is an orthogonal set and this implies $\beta = 0$ and $\alpha_j = \beta = 0 \quad j = 0, 1, \cdots, m-2$. That means that $\alpha_j = 0 \quad j = 0, 1, \cdots, m-2$. Hence, $s_j^\star(t) \quad j = 0, 1, \cdots, m-2$ are linearly independent. We have proved that the new set spans a space of dimension $m-1$.

3. It is easy to show that n-tuple corresponding to $s_j^\star$ is $\sqrt{\mathcal{E}_s}\frac{m-1}{m}$ at position j and $\frac{\sqrt{\mathcal{E}_s}}{m}$ at all other positions. Clearly $||\boldsymbol{s}_j^\star||^2 = (m-1)\frac{\mathcal{E}_s}{m^2} + \frac{\mathcal{E}_s}{m^2}(m-1)^2 = \mathcal{E}_s(1 - \frac{1}{m})$. This is independent of j so the average energy is also $\mathcal{E}_s(1 - \frac{1}{m})$.

Problem 5. *(m-ary Frequency shift Keying)*

1. Orthogonality requires $\int_0^T \cos(2\pi(f_c + i\Delta f)t)\cos(2\pi(f_c + j\Delta f)t)dt = 0$ for every $i \neq j$. Using the trigonometric identity $\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha - \beta)$, an equivalent condition is $\frac{1}{2}\int_0^T[\cos(2\pi(i-j)\Delta f t) + \cos(2\pi(2f_c + (i+j)\Delta f)t)]dt = 0$. Integrating we obtain $\frac{\sin(2\pi(i-j)\Delta f T)}{2\pi(i-j)\Delta f} + \frac{\sin(2\pi(2f_c + (i+j)\Delta f)T)}{2\pi(2f_c+(i+j)\Delta f)} = 0$. As $f_c T$ is assumed to be an integer, the result can be simplified to $\frac{\sin(2\pi(i-j)\Delta f T)}{2\pi(i-j)\Delta f} + \frac{\sin(2\pi(i+j)\Delta f T)}{2\pi(2f_c+(i+j)\Delta f)} = 0$. As i and j are integer this result is zeros for $i \neq j$ if and only if $2\pi\Delta f T$ is an integer multiple of π. Hence, we obtain the minimum value of Δf if $2\pi\Delta f T = \pi$ which gives $\Delta f = \frac{1}{2T}$.

2. Proceeding similarly we will have orthogonality if and only if $\frac{\sin(2\pi(i-j)\Delta fT+\theta_i-\theta_j)-\sin(\theta_i-\theta_j)}{2\pi(i-j)\Delta f}+\frac{\sin(2\pi(i+j)\Delta fT+\theta_i+\theta_j)-\sin(\theta_i+\theta_j)}{2\pi(2f_c+(i+j)\Delta f)}=0$. In this case we see that both parts become zero if and only if $2\pi\Delta fT$ is an even multiple of π which means that the smallest Δf is $\Delta f=\frac{1}{T}$ which is twice the minimum frequency separation needed in the previous part. Hence, the cost of phase uncertainty is a bandwidth expansion by a factor of 2.

3. The condition we obtained for the orthogonality in the first part consist of two terms as follows $\int_0^T[\cos(2\pi(i-j)\Delta ft)+\cos(2\pi(2f_c+(i+j)\Delta f)t)]dt=0$. We saw that if f_cT is exactly an integer number then with have orthogonality with $\Delta f=\frac{1}{2T}$. Now assume that $f_c >> M\Delta f$ in this case the integral value will be $\frac{\sin(2\pi(2f_c+(i+j)\Delta f)T}{2\pi(2f_c+(i+j)\Delta f}$ which its absolute value is always less that $\frac{1}{2\pi(2f_c+(i+j)\Delta f}$ which approaches zero as f_c becomes bigger and bigger. So if we choose $\Delta f=\frac{1}{2T}$ and take $f_c >> m\Delta f$ then we will have approximately orthogonality. In a similar way when we have a random phase shift then we can choose $\Delta f=\frac{1}{T}$ and take $f_c >> m\Delta f$ to have orthogonality.

4. Integrating $\boldsymbol{s}_i(t)^2$ over $[0,T]$ we obtain $A^2\times\frac{2}{T}\times\frac{1}{2}\times T=A^2$ which holds for every i. Hence, the mean energy of the constellation is A^2 but this energy is transmitted during $[0,T]$ so the mean power will be $\frac{A^2}{T}$ which is independent of k.

5. We have M signals separated by Δf. The approximate bandwidth is $m\Delta f$. This means bandwidth $\frac{2^k}{2T}$ in the former case, without random phase shift, and bandwidth $\frac{2^k}{T}$ in the latter case in which we have a random phase shift.

6. Practical systems have a constant B and a T which grows linearly with k. Even if we let T grow linearly with k, in the system considered here, B grows exponentially with k. This is not practical.