ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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Principles of Digital Communications:	Assignment date: Mar 21, 2011
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Solution of Homework 6

Problem 1. *(Uniform Polar To Cartesian)*

1. At first look it seems that probability is uniformly distributed over the disk but in the next part we will show that this is not true.

2. We know that R is uniformly distributed in $[0, 1]$ and Φ is uniformly distributed in $[0, 2\pi[$ so we have $f_R(r) = 1$ if $0 \le r \le 1$ and $f_\Phi(\phi) = \frac{1}{2\pi}$ if $0 \le \phi < 2\pi$. As these two random variables are independent we have $f_{R,\Phi}(r, \phi) = \begin{cases} \frac{1}{2\pi} & 0 \le r \le 1 \text{ and } 0 \le \phi < 2\pi \\ 0 & \text{otherwise.} \end{cases}$
   It can be easily shown that the Jacobian determinant is $r = \sqrt{x^2 + y^2}$ and so the probability distribution in Cartesian coordinates will be $f_{X,Y}(x, y) = \begin{cases} \frac{1}{2\pi\sqrt{x^2+y^2}} & x^2 + y^2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$

3. We see that the probability distribution is not distributed uniformly. This makes sense because rings of equal width have the same probability but not the same area. Hence the density is not constant. Actually as the radius of the ring increases, its area increases proportional to the radius, so the distribution decreases proportional to the inverse of the radius.

Problem 2. *(Gaussian Random Variables)*

1. (a) The joint density of X and Y is $f_{X,Y}(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$. Using polar change of variable we have

$$\begin{aligned} f_{R,\Theta}(r, \theta) &= f_{X,Y}(x, y) \mid_{x=r\cos\theta, y=r\sin\theta} |J(r, \theta)| \\ &= \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \end{aligned}$$

   (b) We see that we can factorize the joint probability density as $f_{R,\Theta}(r, \theta) = \left(\frac{1}{2\pi}\right) \left(\frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\right)$, which implies that R and Θ are independent.

(c) From the factorization $f_\Theta(\theta) = \frac{c}{2\pi}$ where c is the normalizing constant. As $\theta \in [0, 2\pi)$ we obtain that $c = 1$. This implies that Θ is uniformly distributed over $[0, 2\pi)$.

(d) Any rotation of X and Y just changes Θ. As Θ is uniformly distributed, any rotation of Θ is again uniformly distributed over $[0, 2\pi)$. Hence, rotation doesn't change the joint distribution of X and Y.

(e) We see that the joint probability density of X and Y just depends on $r = \sqrt{x^2 + y^2}$ which is invariant under rotation. This implies the invariance of joint density under rotation.

2. (a) $\phi(r) = f_{X,Y}(x, y) = f_X(x)f_Y(y) = g(x)g(y)$ where we used the i.i.d property of X and Y.

   (b) Taking derivative w.r.t x we obtain $g'(x)g(y) = \phi'(r)\frac{dr}{dx} = \phi'(r)\frac{x}{r}$. Dividing both sides by $x\phi(r)$ we obtain $\frac{g'(x)}{xg(x)} = \frac{\phi'(r)}{r\phi(r)}$. By symmetry we have

$$\frac{g'(x)}{xg(x)} = \frac{\phi'(r)}{r\phi(r)} = \frac{g'(y)}{yg(y)} \tag{1}$$

   (c) In (1) the left side is a function of x, whereas the right side is a function of y. Hence the identity holds if both are equal to a constant λ. In other words,

$$\frac{g'(x)}{xg(x)} = \frac{\phi'(r)}{r\phi(r)} = \frac{g'(y)}{yg(y)} = \lambda. \tag{2}$$

   (d) Integrating the identity (2), we obtain

$$\frac{g'(x)}{xg(x)} = \lambda \Rightarrow \frac{g'(x)}{g(x)} = \lambda x \Rightarrow \log\left(\frac{g(x)}{c}\right) = \lambda\frac{x^2}{2} \Rightarrow g(x) = c\, e^{\lambda\frac{x^2}{2}}$$

   which has the form of a **zero mean** Gaussian distribution.

Problem 3. *(Real-Valued Gaussian Random Variables)*

1. It is sufficient to find the marginal distribution of X and show that it is also Gaussian. If we define $\rho \triangleq \frac{\sigma_{XY}}{\sigma_X\sigma_Y}$ then we can write joint distribution as $f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp(-\frac{1}{2(1-\rho^2)}(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}))$. Now put the y term in square form and so we have $f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp(-\frac{x^2}{2\sigma_X^2})\exp(-\frac{1}{2(1-\rho^2)\sigma_Y^2}(y - \frac{\rho\sigma_Y x}{\sigma_X})^2)$. If we integrate from $-\infty$ up to ∞ with respect to y and use the formula $\int exp(-\alpha x^2) = \sqrt{\frac{\pi}{\alpha}}$ we will have $f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp(-\frac{x^2}{2\sigma_X^2})$ which is the Gaussian distribution with mean 0 and variance σ_X^2. And using a similar method we can show that $Y \sim \mathcal{N}(0, \sigma_Y^2)$.

2. Using the notes we know that if X, Y are jointly Gaussian random variables then any linear combination $\alpha X + \beta Y$ for every α and β will be Gaussian distributed. If we set $\alpha = 1$ and $\beta = 0$ then we obtain X which should be Gaussian distributed and similarly for $\alpha = 0$ and $\beta = 1$.

3. If X and Y are independent then $f_{X,Y}(x, y) = f_X(x) f_Y(y)$ and so putting Gaussian distribution formula into the expression we will have $f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y} \exp(-(\frac{x^2}{2\sigma_X^2} + \frac{y^2}{2\sigma_Y^2}))$ which has jointly Gaussian distribution form and so X and Y will be jointly Gaussian random variables.

4. As an example suppose that E is a random variable that takes values $+1$ and -1 with equal probability and X is a zero mean and unit variance Gaussian random variable and independent of E. In this case we consider the random variables $Z = EX$ and $W = X$ then it can be shown that Z and W have the same distribution as X but are not independent because if W takes value a then $Z = Ea$ which can only take $\pm a$. If W and Z are jointly Gaussian then $W + Z$ should be also Gaussian but $W + Z = (1 + E)X$ which takes value 0 with probability $\frac{1}{2}$ and since $W + Z$ is not Gaussian, W and Z can't be jointly Gaussian.

5. X and Y are Gaussian and independent. They are also jointly Gaussian. Hence, any linear combination of X and Y specifically $Z \triangleq X + Y$ will be Gaussian and so it is sufficient to find the mean and the variance of this random variable. $\mu_Z = \mu_X + \mu_Y = 0$ and because X and Y are independent $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$. Hence $X + Y \sim \mathcal{N}(0, \sigma_X^2 + \sigma_Y^2)$.

Problem 4. *(Correlated Noise)*

1. First remember that $\mathbf{Z}$ is a jointly Gaussian random vector. Hence any linear transformation of Z will also be a Gaussian random vector. A Gaussian random vector is characterized by its mean and covariance matrix. Using the hint, we see that if we take $W = AZ$ then W will have zero mean and identity covariance matrix. Also remember that for a jointly Gaussian random vector uncorrelatedness implies independence and so the components of $W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ are independent. Notice also that determinant of A is not zero. Hence the transformation is reversible.

2. The new hypothesis testing problem will be $H = i : \ \hat{\mathbf{Y}} = \hat{\mathbf{s}}_i + \mathbf{W} \quad i = 1, 2, 3, 4$ where $\hat{\mathbf{s}}_i = B\mathbf{s}_i$. Hence the new constellation points will be $\mathbf{s}_0 \to \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$, $\mathbf{s}_1 \to \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}$, $\mathbf{s}_2 \to \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}$ and $\mathbf{s}_3 \to \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$

3. The probability of error is minimized by minimum distance decoder. Unfortunately the boundary regions are not orthogonal to one another and this makes it more difficult to

compute the probability of error. However from the drawing of new constellation points we see immediately that $\mathcal{R}_0^c = B_{0,1} \cup B_{0,2} \cup B_{0,3}$. Hence, using the union bound $P_e(0) \leq Q(\frac{||s_0 - s_3||}{2\sigma}) + Q(\frac{||s_0 - s_2||}{2\sigma}) + Q(\frac{||s_0 - s_1||}{2\sigma})$ and by symmetry similar results hold for other hypotheses. Hence, using the fact that $\sigma = 1$ we have $P_e(0) = P_e(2) < Q(\frac{\sqrt{13}}{4}) + Q(1) + Q(\frac{\sqrt{5}}{4})$ and $P_e(1) = P_e(3) < Q(\frac{\sqrt{13}}{4}) + Q(\frac{\sqrt{5}}{4}) + Q(\frac{\sqrt{5}}{2})$. And assuming equiprobable canstellation points we have $P_e = \frac{P_e(0) + P_e(1)}{2} < Q(\frac{\sqrt{13}}{4}) + Q(\frac{\sqrt{5}}{4}) + \frac{Q(\frac{\sqrt{5}}{2}) + Q(1)}{2}$.