ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Principles of Digital Communications:	Assignment date: Mar 21, 2012
Summer Semester 2012	Due date: Mar 28, 2012

Homework 6

Reading Part for next Wednesday: From Appendix 2.E (Inner Product Spaces) till end of Section 3.3 (Observables and Sufficient Statistics).

Problem 1. *(Uniform Polar To Cartesian)*
Let R and Φ be independent random variables. R is distributed uniformly over the unit interval, Φ is distributed uniformly over the interval $[0, 2\pi)$.[1]

1. Interpret R and Φ as the polar coordinates of a point in the plane. It is clear that the point lies inside (or on) the unit circle. Is the distribution of the point uniform over the unit disk? Take a guess!

2. Define the random variables

$$\begin{aligned} X &= R\cos\Phi \\ Y &= R\sin\Phi. \end{aligned}$$

   Find the joint distribution of the random variables X and Y using the Jacobian determinant.

3. Does the result of part (2) support or contradict your guess from part (1)? Explain.

Problem 2. *(Gaussian Random Variables)*

1. Assume that X and Y are i.i.d. $\mathcal{N}(0, \sigma^2)$ random variables.

[1] This notation means: 0 is included, but 2π is excluded. It is the current standard notation in the anglo-saxon world. In the French world, the current standard for the same thing is $[0, 2\pi[$.

(a) Use the polar change of variables to find the joint probability density of (R, Θ).
Hint: We have the following transformation $X = R\cos(\Theta), Y = R\sin(\Theta)$.

(b) Show that R and Θ independent from each other.

(c) Show that Θ has a uniform distribution.

(d) Use the independence and uniformity assumption to show that any rotation of (X, Y) has the same distribution as (X, Y).

(e) Argue that this happens because of the circular symmetry of the joint distribution of X and Y.

2. In this part we are going to prove the reverse part. Assume that X and Y are i.i.d. random variables and their joint density has circular symmetry. In other words, $f_{X,Y}(x, y) = \phi(r)$ where $r = \sqrt{x^2 + y^2}$.

(a) Show that $\phi(r) = g(x)g(y)$, where g is the probability density of X and Y.

(b) Take the derivative with respect to x and y and simplify it to obtain

$$\frac{g'(x)}{xg(x)} = \frac{\phi'(r)}{r\phi(r)} = \frac{g'(y)}{yg(y)}. \tag{1}$$

(c) Show that identity (1) holds provided that all of the three parts are equal to the same constant. In other words,

$$\frac{g'(x)}{xg(x)} = \frac{\phi'(r)}{r\phi(r)} = \frac{g'(y)}{yg(y)} = \lambda, \tag{2}$$

where λ is a constant.

(d) Show that (2) implies that

$$g(x) \propto e^{\frac{\lambda x^2}{2}}.$$

In other words, X and Y are **zero mean Gaussian** random variables.

Problem 3. *(Real-Valued Gaussian Random Variables)*
For the purpose of this problem, two zero-mean real-valued Gaussian random variables X and Y are called *jointly* Gaussian if and only if their joint density is

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2}\begin{pmatrix} x, & y \end{pmatrix} \Sigma^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\right), \tag{3}$$

where (for zero-mean random vectors) the so-called *covariance matrix* Σ is

$$\Sigma = E\left[\begin{pmatrix} X \\ Y \end{pmatrix}(X, Y)\right] = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}. \tag{4}$$

1. Show that if X and Y are zero-mean jointly Gaussian random variables, then X is a zero-mean Gaussian random variable, and so is Y.

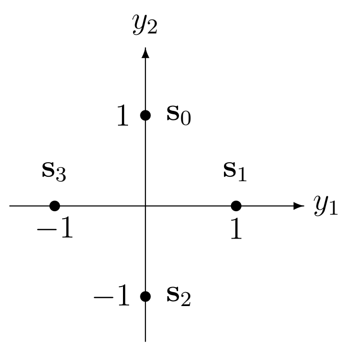
2. How does your answer change if you use the definition of jointly Gaussian random variables given in these notes?

3. Show that if X and Y are independent zero-mean Gaussian random variables, then X and Y are zero-mean jointly Gaussian random variables.

4. However, if X and Y are Gaussian random variables but *not* independent, then X and Y are not necessarily jointly Gaussian. Give an example where X and Y are Gaussian random variables, yet they are *not* jointly Gaussian.

5. Let X and Y be independent Gaussian random variables with zero mean and variance σ_X^2 and σ_Y^2, respectively. Find the probability density function of $Z = X + Y$.

Problem 4. *(Correlated Noise)*
Consider the following communication problem. The message is represented by a uniformly distributed random variable H taking values in $\{0, 1, 2, 3\}$. When $H = i$ we send $\mathbf{s}_i$ where $\mathbf{s}_0 = (0, 1)^T$, $\mathbf{s}_1 = (1, 0)^T$, $\mathbf{s}_2 = (0, -1)^T$, $\mathbf{s}_3 = (-1, 0)^T$ (see the figure below).



When $H = i$, the receiver observes the vector $\mathbf{Y} = \mathbf{s}_i + \mathbf{Z}$, where $\mathbf{Z}$ is a zero-mean Gaussian random vector whose covariance matrix is $\Sigma = \left(\begin{smallmatrix} 4 & 2 \\ 2 & 5 \end{smallmatrix}\right)$.

1. In order to simplify the decision problem, we transform $\mathbf{Y}$ into $\hat{\mathbf{Y}} = B\mathbf{Y} = B\mathbf{s}_i + B\mathbf{Z}$, where B is a 2-by-2 invertible matrix, and use $\hat{\mathbf{Y}}$ as a sufficient statistic. Find a B such that $B\mathbf{Z}$ is a zero-mean Gaussian random vector with independent and identically distributed components. Hint: If $A = \frac{1}{4}\left(\begin{smallmatrix} 2 & 0 \\ -1 & 2 \end{smallmatrix}\right)$, then $A\Sigma A^T = I$, with $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

2. Formulate the new hypothesis testing problem that has $\hat{\mathbf{Y}}$ as the observable and depict the decision regions.

3. Give an upper bound to the error probability in this decision problem.