ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Principles of Digital Communications: Summer Semester 2012

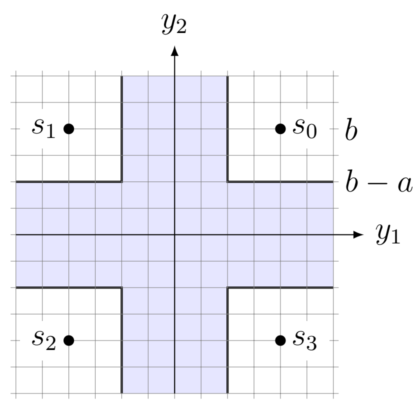
Assignment date: March 14, 2012
Due date: March 21, 2012

Homework 5

Project: Now where you have formed the groups for your project, we start slowly with some of the necessary ingredients which you will need to build your communications system. Decide on what programming language you will use. Then find out how to address both the loudspeaker as well as the microphone of your laptop. Write a small routine which can play a sound wave consisting of a 400 Hz sine wave for exactly one second. Write another routine which records and stores the recoding in a file.

Reading Part for the next week: Appedix 2.B (From Densities After One-To-One Differentiable Transformations) and Appendix 2.C (Gaussian Random Vectors).

Problem 1. *(QAM with Erasure)*
Consider a QAM receiver that outputs a special symbol called "erasure" and denoted by δ whenever the observation falls in the shaded area shown in the figure. Assume that $\boldsymbol{s}_0$ is transmitted and that $\boldsymbol{Y} = \boldsymbol{s}_0 + \boldsymbol{N}$ is received where $\boldsymbol{N} \sim \mathcal{N}(0, \sigma^2 I_2)$. Let P_{0i}, $i = 0, 1, 2, 3$ be the probability that the receiver outputs $\hat{H} = i$ and let$P_{0\delta}$ be the probability that it outputs δ. Determine P_{00}, P_{01}, P_{02}, P_{03} and $P_{0\delta}$.



Comment: When does the observer makes sence? If we choose b small enough we can make sure that the probability of the error is very small (we say that an error occurred if $\hat{H} = i,\ i \in \{0, 1, 2, 3\}$ and $H \neq \hat{H}$). When $\hat{H} = \delta$, the receiver can ask for a retransmission of H. This requires a feedback channel from the receiver to the sender. In most practical applications such a feedback channel is available.

Problem 2. *(Gaussian Hypothesis Testing)*
Assume that we have an n-dimensional observation $Y = (Y_1, ..., Y_n)$. Under H_i , $Y \sim N(\mu_i, \sigma_i^2 I_n)$, $i = 0, 1$, where μ_0 and μ_1 are n-dimensional vectors and I_n is the identity matrix of dimension n. Also suppose that H_0 and H_1 are equiprobable.

1. (a) If $\sigma_1 = \sigma_0 = \sigma$, find the MAP rule and the corresponding decision regions in $\mathbb{R}^n$. How do they look like ?
   (b) Find the error probability as a function of μ_0, μ_1 and σ.

2. (a) If $\mu_0 = \mu_1 = 0$ and $\sigma_0 < \sigma_1$, find the MAP rule and the corresponding decision regions.
   (b) For the simple case of $n = 2$, find the error probability using the following steps :
      i. Show that if Y_1 and Y_2 are $N(0, \sigma^2)$ then $Y_1^2 + Y_2^2$ is an exponential random variable with parameter $\frac{1}{2\sigma^2}$. **Hint:** an exponential random variable z with parameter λ has the following density:

$$f_z(z) = \begin{cases} \lambda e^{-\lambda z} & \text{if } z \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

      ii. Find the conditional and the mean error probability as a function of σ_0 and σ_1.
      iii. Show that as $\rho = \frac{\sigma_1}{\sigma_0} \to \infty$, the error probability goes to zero.

Problem 3. *(Repeat Codes and Bhattacharyya Bound)*
Consider two equally likely hypotheses. Under hypothesis $H = 0$, the transmitter sends $s_0 = (1, \ldots, 1)$ and under $H = 1$ it sends $s_1 = (-1, \ldots, -1)$. The channel model is the AWGN with variance σ^2 in each component. Recall that the probability of error for a ML receiver that observes the channel output Y is

$$P_e = Q\left(\frac{\sqrt{N}}{\sigma}\right).$$

Suppose now that the decoder has access *only* to the sign of Y_i, $1 \leq i \leq N$. That is, the observation is

$$W = (W_1, \ldots, W_N) = (sign(Y_1), \ldots, sign(Y_N)). \tag{1}$$

1. Determine the MAP decision rule based on the observation W. Give a simple sufficient statistic, and draw a diagram of the optimal receiver.

2. Find the expression for the probability of error $\tilde{P}_e$ of the MAP decoder that observes W. You may assume that N is odd.

3. Your answer to *(b)* contains a sum that cannot be expressed in closed form. Express the Bhattacharyya bound on $\tilde{P}_e$.

4. For $N = 1, 3, 5, 7$, find the numerical values of P_e, $\tilde{P}_e$, and the Bhattacharyya bound on $\tilde{P}_e$.

Problem 4. *(Tighter Union Bhattacharyya Bound: Binary Case)*
In this problem we derive a tighter version of the *Union Bhattacharyya Bound* for binary hypotheses. Let

$$\begin{aligned} H = 0 &: \; Y \sim f_{Y|H}(y|0) \\ H = 1 &: \; Y \sim f_{Y|H}(y|1). \end{aligned}$$

The MAP decision rule is

$$\hat{H}(y) = \arg\max_i P_H(i) f_{Y|H}(y|i),$$

and the resulting probability of error is

$$Pr\{e\} = P_H(0) \int_{\mathcal{R}_1} f_{Y|H}(y|0)dy + P_H(1) \int_{\mathcal{R}_0} f_{Y|H}(y|1)dy.$$

1. Argue that
$$Pr\{e\} = \int_y \min\left\{P_H(0) f_{Y|H}(y|0), P_H(1) f_{Y|H}(y|1)\right\} dy.$$

2. Prove that for $a, b \geq 0,\; \min(a, b) \leq \sqrt{ab} \leq \frac{a+b}{2}$. Use this to prove the tighter version of *Bhattacharyya Bound*, i.e,
$$Pr\{e\} \leq \frac{1}{2} \int_y \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)} dy.$$

3. Compare the above bound to the one derived in class when $P_H(0) = \frac{1}{2}$. How do you explain the improvement by a factor $\frac{1}{2}$?

Problem 5. *(Applying the Tight Bhattacharyya Bound)*
As an application of the tight Bhattacharyya bound, consider the following binary hypothesis testing problem

$$\begin{aligned} H = 0 &: \; Y \sim \mathcal{N}(-a, \sigma^2) \\ H = 1 &: \; Y \sim \mathcal{N}(+a, \sigma^2) \end{aligned}$$

where the two hypotheses are equiprobable.

1. Use the *Tight Bhattacharyya Bound* to derive a bound on P_e.

2. We know that the probability of error for this binary hypothesis testing problem is $Q(\frac{a}{\sigma}) \leq \frac{1}{2}\exp\left\{-\frac{a^2}{2\sigma^2}\right\}$, where we have used the result $Q(x) \leq \frac{1}{2}\exp\left\{-\frac{x^2}{2}\right\}$ derived in lecture 1. How do the two bounds compare? Are you surprised (and why)?

Problem 6. *(Bhattacharyya Bound for DMCs)*
Consider a Discrete Memoryless Channel (DMC). This is a channel model described by an input alphabet $\mathcal{X}$, an output alphabet $\mathcal{Y}$ and a transition probability[1] $P_{Y|X}(y|x)$. When we use this channel to transmit an n-tuple $x \in \mathcal{X}^n$, the transition probability is

$$P_{Y|X}(y|x) = \prod_{i=1}^{n} P_{Y|X}(y_i|x_i).$$

So far we have come across two DMCs, namely the BSC (Binary Symmetric Channel) and the BEC (Binary Erasure Channel). The purpose of this problem is to realize that for DMCs, the *Bhattacharyya Bound* takes on a simple form, in particular when the channel input alphabet $\mathcal{X}$ contains only two letters.

1. Consider a source that sends s_0 when $H = 0$ and s_1 when $H = 1$. Justify the following chain of inequalities.

$$\begin{aligned}
P_e &\overset{(a)}{\leq} \sum_y \sqrt{P_{Y|X}(y|s_0)P_{Y|X}(y|s_1)} \\
&\overset{(b)}{\leq} \sum_y \sqrt{\prod_{i=1}^{n} P_{Y|X}(y_i|s_{0i})P_{Y|X}(y_i|s_{1i})} \\
&\overset{(c)}{=} \sum_{y_1,\ldots,y_n} \prod_{i=1}^{n} \sqrt{P_{Y|X}(y_i|s_{0i})P_{Y|X}(y_i|s_{1i})} \\
&\overset{(d)}{=} \sum_{y_1} \sqrt{P_{Y|X}(y_1|s_{01})P_{Y|X}(y_1|s_{11})} \ldots \sum_{y_n} \sqrt{P_{Y|X}(y_n|s_{0n})P_{Y|X}(y_n|s_{1n})} \\
&\overset{(e)}{=} \prod_{i=1}^{n} \sum_y \sqrt{P_{Y|X}(y|s_{0i})P_{Y|X}(y|s_{1i})} \\
&\overset{(f)}{=} \prod_{a\in\mathcal{X}, b\in\mathcal{X}, a\neq b} \left(\sum_y \sqrt{P_{Y|X}(y|s_{0i})P_{Y|X}(y|s_{1i})} \right)^{n(a,b)}.
\end{aligned}$$

where $n(a,b)$ is the number of positions i in which $s_{0i} = a$ and $s_{1i} = b$.

[1]Here we are assuming that the output alphabet is discrete. Otherwise we need to deal with densities instead of probabilities.

2. The Hamming distance $d_H(s_0, s_1)$ is defined as the number of positions in which s_0 and s_1 differ. Show that for a binary input channel, i.e, when $\mathcal{X} = \{a, b\}$, the *Bhattacharyya Bound* becomes

$$P_e \leq z^{d_H(s_0,s_1)},$$

where

$$z = \sum_y \sqrt{P_{Y|X}(y|a)P_{Y|X}(y|b)}.$$

Notice that z depends only on the channel whereas its exponent depends only on s_0 and s_1.

3. What is z for:

(a) The binary input Gaussian channel described by the densities

$$\begin{aligned} f_{Y|X}(y|0) &= \mathcal{N}(-\sqrt{E}, \sigma^2) \\ f_{Y|X}(y|1) &= \mathcal{N}(\sqrt{E}, \sigma^2). \end{aligned}$$

(b) The Binary Symmetric Channel (BSC) with the transition probabilities described by

$$P_{Y|X}(y|x) = \begin{cases} 1 - \delta, & \text{if } y = x, \\ \delta, & \text{otherwise.} \end{cases}$$

(c) The Binary Erasure Channel (BEC) with the transition probabilities given by

$$P_{Y|X}(y|x) = \begin{cases} 1 - \delta, & \text{if } y = x, \\ \delta, & \text{if } y = E \\ 0, & \text{otherwise.} \end{cases}$$

(d) Consider a channel with input alphabet $\{\pm 1\}$, and output $Y = sign(x + Z)$, where x is the input and $Z \sim \mathcal{N}(0, \sigma^2)$. This is a BSC obtained from quantizing a Gaussian channel used with binary input alphabet. What is the crossover probability p of the BSC? Plot the z of the underlying Gaussian channel (with inputs in $\mathbb{R}$) and that of the BSC. By how much do we need to increase the input power of the quantized channel to match the z of the unquantized channel?