ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Principles of Digital Communications: Summer Semester 2012
Assignment date: Feb 24, 2012
Due date: Feb 29, 2012

Solution of Homework 2

Problem 1. *(Conditioning Technique)*

1. We have

$$
\begin{aligned}
E\{Y|N=k\} &= E\{E\{\sum_{i=1}^{k} X_i|N=k\}\} \\
&= E\{\sum_{i=1}^{k} E\{X_i|N=k\}\} \\
&= E\{\sum_{i=1}^{k} E\{X_i\}\} \\
&= \frac{k}{2},
\end{aligned}
$$

where we used the independence of N and X_i, $i = 1, 2, \ldots, n$. Hence $E\{Y|N\} = \frac{N}{2}$ and using the conditioning we have

$$
\begin{aligned}
E\{Y\} &= E\{E\{Y|N\}\} \\
&= E\{\frac{N}{2}\} \\
&= \frac{1}{2} \times \frac{n+1}{2} \\
&= \frac{n+1}{4}.
\end{aligned}
$$

2. Similar to the previous part we have

$$\begin{aligned}
E\{Y^2|N=k\} &= E\{(\sum_{i=1}^{k} X_i)^2|N=k\} \\
&= E\{(\sum_{i=1}^{k} X_i)^2\} \\
&= E\{X_1^2 + X_1X_2 + \cdots + X_2^2 + X_1X_2 + \dots\} \\
&= E\{\sum_{i=1}^{k} X_i^2 + \sum_{i=1}^{k} \sum_{j=1, j\neq i}^{k} (X_iX_j)\} \\
&= kE\{X_1^2\} + k(k-1)E\{X_1\}^2 \\
&= \frac{k}{3} + \frac{k(k-1)}{4} \\
&= \frac{k^2}{4} + \frac{k}{12}.
\end{aligned}$$

Hence $E\{Y^2|N\} = \frac{N^2}{4} + \frac{N}{12}$. Taking the expectation with respect to N and using the formulas

$$\begin{aligned}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2}, \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6},
\end{aligned}$$

we obtain

$$\begin{aligned}
E\{Y^2\} &= \frac{(n+1)(2n+1)}{24} + \frac{n+1}{24} \\
&= \frac{(n+1)^2}{12},
\end{aligned}$$

and

$$\begin{aligned}
\text{var}(Y) &= E\{Y^2\} - E\{Y\}^2 \\
&= \frac{(n+1)^2}{12} - \frac{(n+1)^2}{16} \\
&= \frac{(n+1)^2}{48}.
\end{aligned}$$

Problem 2. *(Conditioning Technique)*

1. Given $X = x$, Y is uniformly distributed between 0 and x hence

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 \leq y \leq x, 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

2. We use the Bayes rule to find the marginal distribution of Y, $f_Y(y) = f_{Y|X}(y|x)f_X(x)dx$. Notice that for a specific $Y = y$ the value of X is always greater than y hence we have

$$
\begin{aligned}
f_Y(y) &= \int_y^1 f_{Y|X}(y|x)f_X(x)dx \\
&= \int_y^1 \frac{1}{x}dx \\
&= -\log(y),
\end{aligned}
$$

where $y \in [0, 1]$ and 0 otherwise. Using the marginal distribution of Y we can obtain the $E\{Y\}$ as follows

$$
\begin{aligned}
E\{Y\} &= \int y f_Y(y)dy \\
&= -\int_0^1 y\log(y)dy \\
&= -\frac{y^2}{2}(\log(y) - \frac{1}{2})]_0^1 \\
&= \frac{1}{4}.
\end{aligned}
$$

3. We have $E\{Y|X = x\} = \frac{x}{2}$. Using the conditioning on $X = x$ we have

$$
\begin{aligned}
E\{Y\} &= E\{E\{Y|X\}\} \\
&= E\{\frac{X}{2}\} \\
&= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4},
\end{aligned}
$$

which is the same as the previous part.

Problem 3. *(Conditioning Technique)* By Symmetry

$$
\begin{aligned}
E\{Y|X + Y = z\} &= E\{X|X + Y = z\} \\
&= \frac{1}{2}E\{X + Y|X + Y = z\} \\
&= \frac{z}{2},
\end{aligned}
$$

which implies that $E\{X|X + Y\} = \frac{X+Y}{2}$.