ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Principles of Digital Communications: Summer Semester 2012 | Assignment date: May 23, 2012 | Due date: May 30, 2012

Solution of Homework 14

Problem 1. *(Equivalent Representation)*

1. Let $x_E(t) = \alpha(t)\exp[j\beta(t)]$. Then

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)\exp[j2\pi f_0 t]\} \\ &= \sqrt{2}\Re\{\alpha(t)\exp[j\beta(t)]\exp[j2\pi f_0 t]\} \\ &= \sqrt{2}\Re\{\alpha(t)\exp[j(2\pi f_0 t + \beta(t))]\} \\ &= \sqrt{2}\alpha(t)\cos[2\pi f_0 t + \beta(t)]. \end{aligned}$$

   We thus have

$$a(t) = \sqrt{2}\alpha(t) = \sqrt{2}|x_E(t)|$$

   and

$$\theta(t) = \beta(t) = \tan^{-1}\frac{\Im\{x_E(t)\}}{\Re\{x_E(t)\}}.$$

   This shows that a bandpass signal is one that is modulated both in amplitude and in phase.

2. Let $x_E(t) = x_R(t) + jx_I(t)$. Then

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)\exp[j2\pi f_0 t]\} \\ &= \sqrt{2}\Re\{[x_R(t) + jx_I(t)]\exp[j2\pi f_0 t]\} \\ &= \sqrt{2}[x_R(t)\cos(2\pi f_0 t) - x_I(t)\sin(2\pi f_0 t)]. \end{aligned}$$

   Hence we have

$$x_{EI}(t) = \sqrt{2}\Re\{x_E(t)\}$$

   and

$$x_{EQ}(t) = \sqrt{2}\Im\{x_E(t)\}.$$

3. We guess that

$$x_E(t) = \frac{A(t)}{\sqrt{2}} \exp(j\varphi).$$

Indeed

$$\begin{aligned} x(t) &= \sqrt{2}\Re\{x_E(t)\exp(j2\pi f_0 t)\} \\ &= \sqrt{2}\Re\{\frac{A(t)}{\sqrt{2}}\exp(j\varphi)\exp(j2\pi f_0 t)\} \\ &= \Re\{A(t)\exp[j(2\pi f_0 t + \varphi)]\} \\ &= A(t)\cos(2\pi f_0 t + \varphi). \end{aligned}$$

Problem 2. *(Equivalent Baseband Signal)*

1. It is immediate to verify that $\psi_E(t) = \frac{1}{\sqrt{2}}\text{sinc}(\frac{t}{T})$ leads to the $\psi(t) = \sqrt{2}\Re\{\psi_E(t)e^{j2\pi f_0 t}\}$ given in the problem formulation.

2. We have the baseband representation of the signal ψ and filter h so we can do the filtering operation in the equivalent baseband representation. The Fourier transform of $\text{sinc}(\frac{t}{2T})$ is a rectangular shaped pulse in $f \in [-\frac{1}{4T}, \frac{1}{4T}]$ hence the Fourier transform of $\text{sinc}^2(\frac{t}{2T})$ is the convolution of the Fourier transform of $\text{sinc}(\frac{t}{2T})$ with itself which is a triangular shaped waveform in $f \in [-\frac{1}{2T}, \frac{1}{2T}]$ and over this range of frequency the Fourier transform of $\psi_E(t) = \text{sinc}(\frac{t}{T})$ is uniform with amplitude T and a simple drawing in the frequency domain shows that $\psi_E(t) \star h_E(t) = Th_E(t) = \frac{1}{\sqrt{2}}\text{sinc}^2(\frac{t}{2T})$.

3. The equivalent passband signal is $\frac{1}{\sqrt{2}}\text{sinc}^2(\frac{t}{2T})\cos(2\pi f_0 t)$.

Problem 3. *(Bandpass Nyquist Pulses)*

1. From the frequency domain it is seen that $p_{\mathcal{F}}(f) = g(f) \star \{\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)\}$ where $g(f) = \begin{cases} 2 & -\frac{B}{2} \le f \le \frac{B}{2} \\ 0 & \text{otherwise} \end{cases}$ $g(f)$ is the Fourier transform of $2B\ \text{sinc}(Bt)$ and $\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$ is the Fourier transform of $\cos(2\pi f_0 t)$. Hence $p(t) = 2B\ \text{sinc}(Bt)\cos(2\pi f_0 t)$.

2. We have

$$\begin{aligned}\int \psi^2(t)dt =& c^2 \int p^2(t)dt \\ =& c^2 \int p_{\mathcal{F}}^2(f)df \\ =& 2c^2 B = 1\end{aligned}$$

which gives $c = \frac{1}{\sqrt{2B}}$.

3. $\{\psi(t - nT)\}$ forms an orthonormal set for $T = \frac{1}{2B}$ if and only if the Nyquist condition holds. In other words
$$\sum \psi_{\mathcal{F}}^2(f - \frac{n}{T}) = T.$$
A simple drawing of the frequency domain shows that Nyquist condition holds and so $\psi(t)$ is a Nyquist pulse.

4. Once again, we are looking for the values of $f_0 - \frac{B}{2}$ for which the above expression holds. A simple drawing shows that it is the case for $f_0 - \frac{B}{2} = kB$ where k is an integer.