ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Principles of Digital Communications: Assignment date: May 9, 2012

Summer Semester 2012 Due date: May 16, 2012

Solution of Homework 11

Problem 1. It's about the project.

Problem 2. *(Power Spectral Density)*

1. Trivially, we find that when $i = j$, $E[X_i X_j]$ is

$$E[X_i^2] = E[1] = 1.$$

Remember that the D_i are i.i.d Bernoulli($\frac{1}{2}$) random variables. Hence, we find immediately

$$\begin{aligned} E[X_{2n}X_{2n+1}] &= E[D_n D_{n-2} D_n D_{n-1} D_{n-2}] \\ &= E[D_n^2 D_{n-1} D_{n-2}^2] \\ &= E[D_{n-1}] \;=\; 0, \end{aligned}$$

and also

$$\begin{aligned} E[X_{2n}X_{2n+2}] &= E[D_n D_{n-2} D_{n+1} D_{n-1}] \\ &= E[D_n]\, E[D_{n-2}]\, E[D_{n+1}]\, E[D_{n-1}] \;=\; 0 \end{aligned}$$

By continuing this argument we find

$$E[X_i X_j] \;=\; \delta[i-j],$$

thus the sequence X_i is wide-sense stationary with autocorrelation function

$$R_X[i-j] \;=\; E[X_i X_j] = \delta[i-j].$$

2. From the Appendix of Chapter 5, we already know that:

$$\begin{aligned} R_{\tilde{X}}(\tau) &= E[\tilde{X}(t)\tilde{X}(t+\tau)] \\ &= E_s\ E[\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} X_i X_j \psi(t-iT_s-T_0)\psi(t-jT_s-T_0+\tau)] \\ &= E_s \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} E[X_i X_j \psi(t-iT_s-T_0)\psi(t-jT_s-T_0+\tau)] \\ &= E_s \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} E[X_i X_j]\ E[\psi(t-iT_s-T_0)\psi(t-jT_s-T_0+\tau)] \end{aligned}$$

The second expectation is over the random variable T_0, which is uniformly distributed over the interval $[0, T_s)$. Therefore, simply by the definition of expectation, we get

$$E[\psi(t-iT_s-T_0)\psi(t-jT_s-T_0+\tau)] = \int_0^{T_s} \frac{1}{T_s}\psi(t-iT_s-t_0)\psi(t-jT_s-t_0+\tau)dt_0$$

Moreover, since $E[X_i X_j] = \delta[i-j]$, we can write

$$\begin{aligned} R_{\tilde{X}}(\tau) &= E_s \sum_{i=-\infty}^{\infty} E[\psi(t-iT_s-T_0)\psi(t-iT_s-T_0+\tau)] \\ &= E_s \sum_{i=-\infty}^{\infty} \int_0^{T_s} \frac{1}{T_s}\psi(t-iT_s-t_0)\psi(t-iT_s-t_0+\tau)dt_0 \end{aligned}$$

Define $\alpha = t - iT_s - t_0$, and substitute t_0 by α in the integral (note that t is just a constant in our computation):

$$\begin{aligned} R_{\tilde{X}}(\tau) &= E_s \sum_{i=-\infty}^{\infty} \int_{t-iT_s}^{t-(i+1)T_s} \frac{1}{T_s}\psi(\alpha)\psi(\alpha+\tau)(-d\alpha) \\ &= E_s \sum_{i=-\infty}^{\infty} \int_{t-(i+1)T_s}^{t-iT_s} \frac{1}{T_s}\psi(\alpha)\psi(\alpha+\tau)d\alpha. \end{aligned}$$

At this point, the constant t drops out: irrespective of the value of t, the sum of integrals just corresponds to the integral from $-\infty$ to ∞, that is

$$\begin{aligned} R_{\tilde{X}}(\tau) &= \frac{E_s}{T_s}\int_{-\infty}^{\infty} \psi(\alpha)\psi(\alpha+\tau)d\alpha \\ &= E_s R_\psi(\tau) \end{aligned}$$

3. The power spectral density is the Fourier transform of the autocorrelation function. Thus, the power spectral density of the signal $\tilde{X}(t)$ is essentially the same as the power spectral density of the signal $\psi(t)$,

$$S_{\tilde{X}}(f) = E_s S_\psi(f).$$

This means that by choosing $\psi(t)$ appropriately, we can control the bandwidth consumption of our communications scheme.

4. We first compute the autocorrelation function of $\psi(t)$:

$$\begin{aligned} R_\psi(\tau) &= \frac{1}{T_s}\int_{-\infty}^{\infty}\psi(\alpha)\psi(\tau+\alpha)d\alpha \\ &= \begin{cases} \frac{1}{T_s}\left(1-\frac{|\tau|}{T_s}\right), & |\tau| \le T_s \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{1}$$

where we used the fact that $\psi(\alpha)\psi(\tau+\alpha)$ vanishes when $|\tau| > T_s$ and is a rectangle of width $T_s - |\tau|$ and height $\frac{1}{T_s}$ otherwise.

We then take the Fourier transform of $R_\psi(\tau)$ to obtain $S_\psi(f)$. For this, note that

$$\begin{aligned} R_\psi(\tau) &= \frac{1}{T_s}\int_{-\infty}^{\infty}\psi(\alpha)\psi(\alpha+\tau)d\alpha \\ &= \frac{1}{T_s}\int_{-\infty}^{\infty}\psi(u)\psi(\tau-u)du \\ &= \frac{1}{T_s}\,\psi(t)*\psi(-t) \\ &= \frac{1}{T_s}\,\psi(t)*\psi(t), \end{aligned}$$

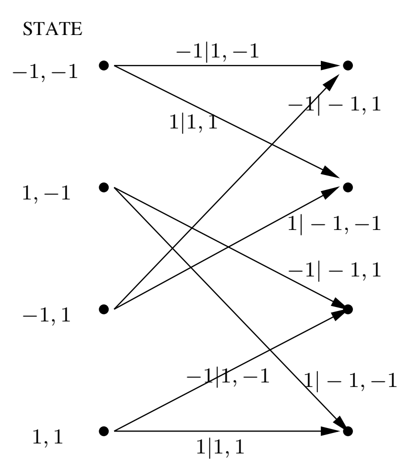
where we used the variable change $u = -\alpha$ and the fact that $\psi(t)$ is an even function. Hence, $R_\psi(\tau)$ is the convolution of ψ with itself. It follows that the Fourier transform of $R_\psi(\tau)$ is the Fourier transform of ψ times itself, *i.e.*,

$$S_\psi(f) = \mathrm{sinc}^2(T_s f).$$

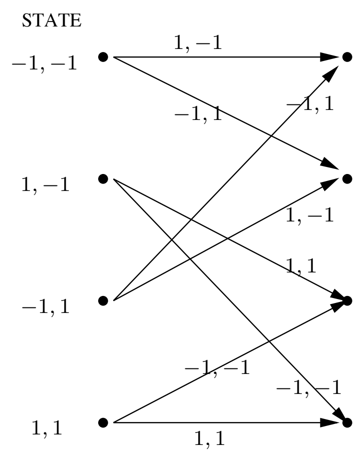
It follows that

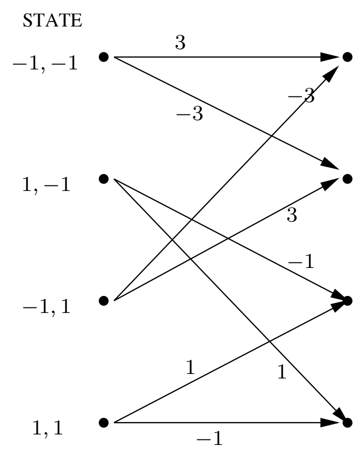
$$S_{\tilde{X}}(f) = E_s\,\mathrm{sinc}^2(T_s f).$$

Problem 3. *(Trellis Section)*
We have the following trellis diagram:

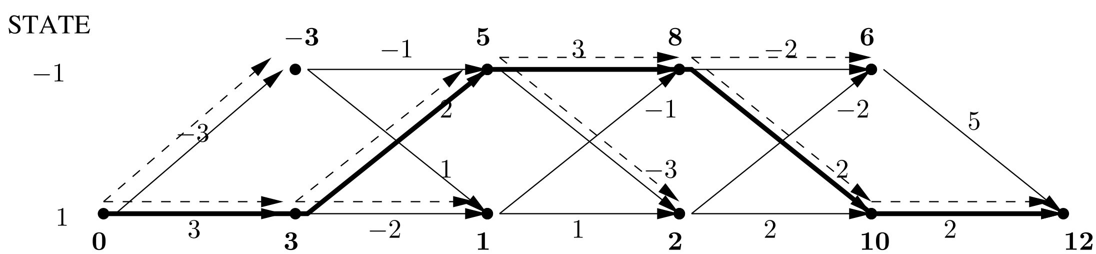


Problem 4. *(Branch Metric)*
For the given trellis diagram, edges are labeled by the output values of the encoder. For example if we are in state $(-1,-1)$ we can go to the state $(-1,-1)$ and the corresponding output of the encoder will be $(1,-1)$. We also know that for AWGN channel the suitable branch metric is the quadratic metric, i.e. if the edges are labeled by the (x_{2n}, x_{2n+1}) and the received values at the output of match filter are (y_{2n}, y_{2n+1}) then the corresponding branch metric is the inner product $x_{2n}y_{2n} + x_{2n+1}y_{2n+1}$ because all of the inputs have the same energy. Hence for the special output $(1,-2)$ in this problem we have the following trellis labelled by the branch metrics.





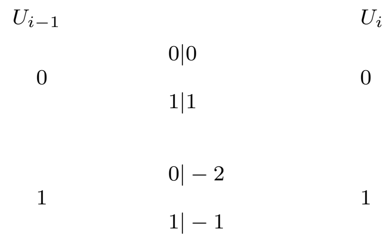
Problem 5. *(Viterbi Algorithm)*
The path in bold corresponds to the ML path to the terminating state. A short arrow (in red) indicates the ML path up to the head of the arrow. Values in bold on the state indicate the value of the ML path until the state.



Problem 6. *(Intersymbol Interference)*

1. We have

$$\begin{aligned} S_i &= \sum_{j=0}^{\infty} U_{i-j} h_j, \qquad i = 1, 2, \dots \\ &= U_i h_0 + U_{i-1} h_1 \\ &= U_i - 2U_{i-1}. \end{aligned}$$



2. We have the following diagram for state transition:

3. We have $\underline{Y} = \underline{S} + \underline{Z}$, where $\underline{Z} = (Z_1, \ldots, Z_6)$ is a sequence of i.i.d. components with $Z_i \sim \mathcal{N}(0, \sigma^2)$. Assuming that $U \sim \text{Bern}(1/2)$, each of the possible sequence $\underline{S}$ is equiprobable. Thus our maximum likelihood decoder is a minimum distance decoder (*Refer to lecture notes*). Hence we have to minimize $||\underline{y} - \underline{s}||^2$ or equivalently, maximize $2 < \underline{y}, \underline{s} > -||\underline{s}||^2$. We thus have $f(\underline{s}, \underline{y}) = \sum_{i=1}^{6} 2y_i s_i - s_i^2$ whose maximization with respect to $\underline{s}$ leads to a maximum likelihood decision on $\underline{S}$.

4. Tracing our path through the trellis, we find that the maximum likelihood estimate of the information sequence is $\underline{U} = (1, 1, 0, 1, 1, 1)$.