Solution of Homework 10

Problem 1. *(Suggestions for the Project)*

Modulation Selection:

1. 
   \[ H(f) = \sum_{k=1}^{K} \alpha_k e^{j2\pi f \tau_k} \]

2. 
   \[ |H(f)| = \sqrt{101 + 20 \cos(2\pi f T)} \]

   The maxima happens at frequencies \( f_n = \frac{n}{T} \) and the minima happens at \( f_n = \frac{2n+1}{2T} \) where \( n \) is an integer.

3. The maximum attenuation is \( |H| = 0.9 \) and occurs at \( f_n = \frac{2n+1}{2T} \) for integer values of \( n \).

4. If the transmitted power is \( P \) in the worst case the received power will be \( |H|_{\text{min}}^2 \times P = 0.81P \). This value must be greater than 1W. This implies that the transmitted power must be greater than \( \frac{100}{81} \approx 1.25W \).

5. Using Fourier analysis we can simply show that 
   \[
   A' = A \times A(f) \\
   f' = f \\
   \phi' = \phi + \theta(f),
   \]

   and we see that the random behavior of the channel doesn’t change the received frequency.
A small thinking show that the MFSK is the best and simplest modulation because it puts the information on a component of the signal which is not effected by the random nature of the channel.

**Random Phase Compensation:**

1. A simple use of the hint gives the result.
2. Simple.
4. 
   \[ p(r_0, r_1, \ldots | H = i) = \int p(r_0, r_1, \ldots | H = i, \phi) \frac{d\phi}{2\pi} \]
   \[ = \frac{1}{\sqrt{(2\pi\sigma^2)^N_s}} \exp(-\frac{\sum_n r_n^2}{2\sigma^2} - \frac{N_sA^2}{4\sigma^2} + \frac{A}{\sigma^2} \sqrt{r_{ic}^2 + r_{is}^2} \cos(\phi)) \frac{d\phi}{2\pi} \]
   \[ = \frac{1}{\sqrt{(2\pi\sigma^2)^N_s}} \exp(-\frac{\sum_n r_n^2}{2\sigma^2} - \frac{N_sA^2}{4\sigma^2}) I_0(\frac{A}{\sigma^2} \sqrt{r_{ic}^2 + r_{is}^2}) \]
5. The first part of the probability distribution \( \frac{1}{\sqrt{(2\pi\sigma^2)^N_s}} \exp(-\frac{\sum_n r_n^2}{2\sigma^2} - \frac{N_sA^2}{4\sigma^2}) \) is the same in both expression \( i = 0, 1 \) so we can simply drop it. Hence the MAP rule can be simply written as
   \[ I_0(\frac{A}{\sigma^2} \sqrt{r_{0c}^2 + r_{0s}^2}) \ll I_0(\frac{A}{\sigma^2} \sqrt{r_{1c}^2 + r_{1s}^2}). \]
6. As \( I_0(x) \) is a monotone increasing function for \( x > 0 \) then the comparison of \( I_0(x) \) values ends up in comparison of the arguments. In other words, the MAP rule can be simplified as
   \[ \sqrt{r_{0c}^2 + r_{0s}^2} \ll \sqrt{r_{1c}^2 + r_{1s}^2}. \]
7. It is easy to check that \( \Re(R[f_i]) = r_{ic} \) and \( \Im(R[f_i]) = r_{is} \). Hence taking the amplitude of the complex number \( R[f_i] \) we obtain \( \sqrt{r_{ic}^2 + r_{is}^2} \). This implies that we can simply implement the receiver in the DFT domain using the FFT algorithm.

**Problem 2. Nyquist Criterion**
To qualify as Nyquist pulses for symbol rate $1/T$, they have to verify the following condition:

$$\sum |\theta_f(f + k/T)|^2 = T. \quad (1)$$

By simply plotting the shifted functions and adding up, one immediately verifies that (a) and (c) are Nyquist pulses of symbol rate $1/T$, but (b) is not.

For (d), we verify in the time domain. We know that $\theta_f(f)$ is a sinc function. Therefore, $\theta(t)$ is a box function. The first zero of the sinc is a $1/T$, which means that the width of the corresponding box function is $T$. So it is immediately clear that $\theta(t)$ is a Nyquist pulse for symbol rate $1/T$ since $\theta(t)$ is orthogonal to $\theta(t - iT)$ for all $i$.

2. For instance, a triangle of height 1 going from $-1$ to 1 in the frequency domain.
3. The block diagram is just the same as always. Suppose that we have a Nyquist pulse \( \theta(t) \) and a sequence of input symbols \( \{X_i\} \). The transmitted signal is (as usual)

\[
x(t) = \sum_{i=\infty}^{\infty} X_i \theta(t - iT_s).
\]

(2)

At the output of the matched filter at time \( jT_s \), we have the value \( Y_j \) determined as follows:

\[
Y_j = \int_{-\infty}^{\infty} R(t) \theta(t - jT_s) dt
\]

(3)

\[
= \sum_{i=\infty}^{\infty} X_i \int_{-\infty}^{\infty} \theta(t - iT_s) \theta(t - jT_s) dt + \int_{-\infty}^{\infty} Z(t) \theta(t - jT_s) dt
\]

(4)

\[
= \sum_{i=\infty}^{\infty} X_i \delta_{ij} + Z_j = X_j + Z_j,
\]

(5)

where \( Z(t) \) is the additive noise process.

Suppose we use a non-Nyquist pulse instead. That is, the pulse is not orthogonal to its shift by multiples of \( T_s \). But then, the value \( Y_j \) computed above will not depend only on \( X_j \), but on other members of the sequence \( \{X_i\} \), too. Thus, we have inter symbol interference. Furthermore, \( \{Z_j\}_{j=-\infty}^{\infty} \) is not an i.i.d. sequence unless \( \{\psi(t - jT)\}_{j=-\infty}^{\infty} \) is an orthogonal sequence.

**Problem 3. (Mixed Questions)**

1. We use the Fourier transform property, namely, multiplication in the time domain is equivalent to convolution in the frequency domain. The Fourier transform of \( \frac{\sin(\pi t)}{\pi} \) is a rectangular pulse with amplitude 1 in \( f \in [-\frac{1}{2}, \frac{1}{2}] \). Hence, the Fourier transform of \( \left(\frac{\sin(\pi t)}{\pi}\right)^2 \) is the convolution of the rectangular pulse with itself which will be a triangular pulse with peak value 1 at \( f = 0 \) and with support \( f \in [-1, 1] \). The Fourier transform of \( \cos(2\pi t) \) is \( \frac{1}{2}\delta(f - 1) + \frac{1}{2}\delta(f + 1) \). Hence the Fourier transform of \( x(t) \) will be the convolution of the Fourier transform of \( \cos(2\pi t) \) with the triangular pulse. A simple drawing of the frequency domain shows that the \( x_F(f) \) is two triangular pulses, both of them have peak value \( \frac{1}{2} \). One of them has the support \( f \in [-2, 0] \) and the other in \( f \in [0, 2] \). As \( x_F \) is a base band signal with bandwidth 4 then the maximum sampling time possible to avoid aliasing is \( \frac{1}{4} \).

2. It is easy to see that \( \int_{0}^{1} s_1(t)s_2(t)dt = 0 \) hence \( s_1 \) and \( s_2 \) are orthogonal and the dimension of the signal set is at least two. Furthermore, we can write \( s_3(t) = \sin^2(\pi t) = \frac{1-\cos(2\pi t)}{2} = \frac{1}{2} s_1(t) - \frac{1}{2} s_2(t) \). Hence the dimensionality of the signal set is two.

3. It is easy to check that \( p(t) \) meets the Nyquist criterion. In other words, a simple drawing of the frequency domain shows that \( \frac{1}{T} \sum |p_F(f - \frac{n}{T})|^2 = 1 \). Hence \( p(t) \) is
orthogonal to its shifted versions by an integer multiple of $T$. In other words \[ \int p(t)p(t - nT)dt = 0, \quad n \neq 0. \] Hence \[ \int p(t)p(t - 3T)dt = 0. \]

**Problem 4. (Power Spectrum: Manchester Pulse)**

1. $r(t)$ is a rectangular pulse in the time domain hence its Fourier transform is sinc. Specifically, $r_F(f) = \frac{\text{sinc}(\frac{\pi f T_s}{2})}{\pi f \sqrt{T_s}}$.

2. We can write $\phi(t)$ as $r(t) \ast (\delta(t - \frac{T_s}{4}) - \delta(t - \frac{3T_s}{4}))$. Hence using the fact that convolution in the time domain is equivalent to multiplication in the frequency domain we have:

   \[
   \phi_F(f) = (\exp(-j2\pi f \frac{T_s}{4}) - \exp(-j2\pi f \frac{3T_s}{4}))r_F(f)
   \]

   \[
   = 2j \exp(-j\pi f T_s) \sin(\frac{\pi f T_s}{2}) \times \frac{\sin(\frac{\pi f T_s}{2})}{\sqrt{T_s} \pi f}
   \]

   \[
   = 2j \exp(-j\pi f T_s) \frac{\sin^2(\frac{\pi f T_s}{2})}{\sqrt{T_s} \pi f}
   \]

   Hence

   \[
   |\phi_F(f)|^2 = 4 \frac{\sin^4(\frac{\pi f T_s}{2})}{T_s \pi^2 f^2}
   \]

3. $\{X_i\}$ are i.i.d random variables with mean zero. Hence $E\{X_iX_j\} = 0, \quad i \neq j$ and $E\{X_i^2\} = E_s$ so we can write $R_X[k] = E_s \delta[k]$ where $\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$. Hence in the summation $\sum R_X[k]e^{-j2\pi kfT_s}$ only the term corresponding to $k = 0$ remains which is equal to $E_s$. Putting together we have:

   \[
   S_X(f) = E_s \frac{\sin^4(\frac{\pi f T_s}{2})}{(\frac{\pi f T_s}{2})^2}
   \]