ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Principles of Digital Communications: Summer Semester 2012

Assignment date: May 2, 2012
Due date: May 9, 2012

Solution of Homework 10

Problem 1. *(Suggestions for the Project)*

Modulation Selection:

1. $$H(f) = \sum_{k=1}^{K} \alpha_k e^{j2\pi f \tau_k}$$

2. $$|H(f)| = \frac{\sqrt{101 + 20\cos(2\pi f T)}}{11}.$$

   The maxima happens at frequencies $f_n = \frac{n}{T}$ and the minima happens at $f_n = \frac{2n+1}{2T}$ where n is an integer.

3. The maximum attenuation is $|H| = 0.9$ and occurs at $f_n = \frac{2n+1}{2T}$ for integer values of n.

4. If the transmitted power is P in the worst case the received power will be $|H|^2_{\min} \times P = 0.81P$. This value must be greater than 1W. This implies that the transmitted power must be greater than $\frac{100}{81} \approx 1.25$W.

5. Using Fourier analysis we can simply show that

   $$A' = A \times A(f)$$
   $$f' = f$$
   $$\phi' = \phi + \theta(f),$$

   and we see that the random behavior of the channel doesn't change the received frequency.

A small thinking show that the MFSK is the best and simplest modulation because it puts the information on a component of the signal which is not effected by the random nature of the channel.

Random Phase Compensation:

1. A simple use of the hint gives the result.

2. Simple.

3. Simple.

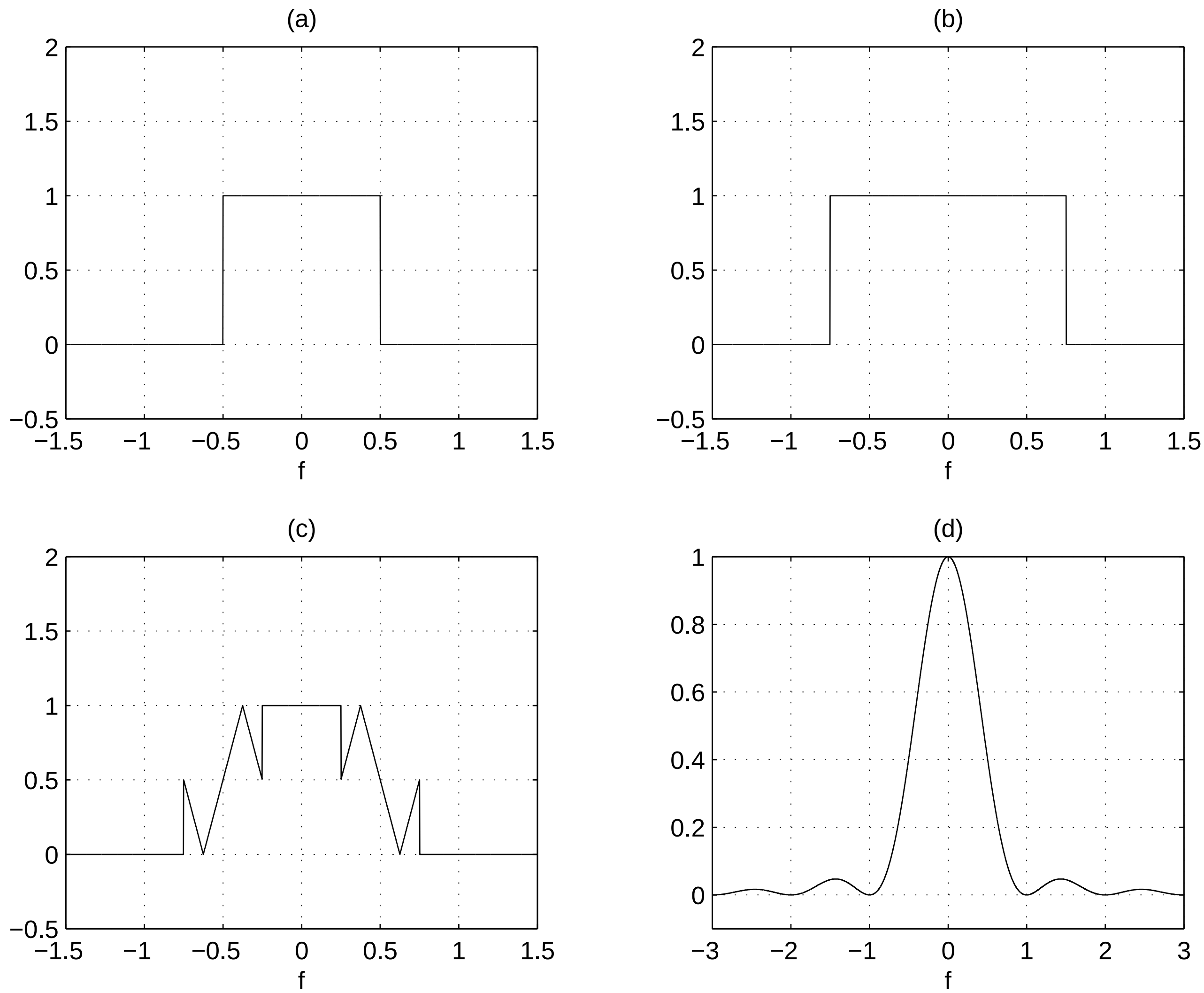
4.
$$\begin{aligned} p(r_0, r_1, \ldots | H = i) &= \int p(r_0, r_1, \ldots | H = i, \phi) \frac{d\phi}{2\pi} \\ &= \frac{1}{\sqrt{(2\pi\sigma^2)^{N_s}}} \exp(-\frac{\sum_n r_n^2}{2\sigma^2} - \frac{N_s A^2}{4\sigma^2} + \frac{A}{\sigma^2}\sqrt{r_{ic}^2 + r_{is}^2}\cos(\phi))\frac{d\phi}{2\pi} \\ &= \frac{1}{\sqrt{(2\pi\sigma^2)^{N_s}}} \exp(-\frac{\sum_n r_n^2}{2\sigma^2} - \frac{N_s A^2}{4\sigma^2}) I_0(\frac{A}{\sigma^2}\sqrt{r_{ic}^2 + r_{is}^2}) \end{aligned}$$

5. The first part of the probability distribution $\frac{1}{\sqrt{(2\pi\sigma^2)^{N_s}}} \exp(-\frac{\sum_n r_n^2}{2\sigma^2} - \frac{N_s A^2}{4\sigma^2})$ is the same in both expression $i = 0, 1$ so we can simply drop it. Hence the MAP rule can be simply written as
$$I_0(\frac{A}{\sigma^2}\sqrt{r_{0c}^2 + r_{0s}^2}) <> I_0(\frac{A}{\sigma^2}\sqrt{r_{1c}^2 + r_{1s}^2}).$$

6. As $I_0(x)$ is a monotone increasing function for $x > 0$ then the comparison of $I_0(x)$ values ends up in comparison of the arguments. In other words, the MAP rule can be simplified as
$$\sqrt{r_{0c}^2 + r_{0s}^2}) <> \sqrt{r_{1c}^2 + r_{1s}^2}.$$

7. It is easy to check that $\Re(R[f_i]) = r_{ic}$ and $\Im(R[f_i]) = r_{is}$. Hence taking the amplitude of the complex number $R[f_i]$ we obtain $\sqrt{r_{ic}^2 + r_{is}^2}$. This implies that we can simply implement the receiver in the DFT domain using the FFT algorithm.

Problem 2. *Nyquist Criterion*

1. 

To qualify as Nyquist pulses for symbol rate $1/T$, they have to verify the following condition:

$$\sum |\theta_{\mathcal{F}}(f + k/T)|^2 \quad = \quad T. \tag{1}$$

By simply plotting the shifted functions and adding up, one immediately verifies that *(a)* and *(c)* are Nyquist pulses of symbol rate $1/T$, but *(b)* is not.

For *(d)*, we verify in the time domain. We know that $\theta_{\mathcal{F}}(f)$ is a sinc function. Therefore, $\theta(t)$ is a box function. The first zero of the sinc is a $1/T$, which means that the width of the corresponding box function is T. So it is immediately clear that $\theta(t)$ is a Nyquist pulse for symbol rate $1/T$ since $\theta(t)$ is orthogonal to $\theta(t - iT)$ for all i.

2. For instance, a triangle of height 1 going from -1 to 1 in the frequency domain.

3. The block diagram is just the same as always. Suppose that we have a Nyquist pulse $\theta(t)$ and a sequence of input symbols $\{X_i\}$. The transmitted signal is (as usual)

$$x(t) \;=\; \sum_{i=-\infty}^{\infty} X_i \theta(t - iT_s). \tag{2}$$

At the output of the matched filter at time jT_s, we have the value Y_j determined as follows:

$$\begin{aligned} Y_j &= \int_{-\infty}^{\infty} R(t)\theta(t - jT_s)dt & (3)\\ &= \sum_{i=-\infty}^{\infty} X_i \int_{-\infty}^{\infty} \theta(t - iT_s)\theta(t - jT_s)dt \;+\; \int_{-\infty}^{\infty} Z(t)\theta(t - jT_s)dt & (4)\\ &= \sum_{i=-\infty}^{\infty} X_i\delta_{ij} \;+\; Z_j \;=\; X_j + Z_j, & (5) \end{aligned}$$

where $Z(t)$ is the additive noise process.

Suppose we use a non-Nyquist pulse instead. That is, the pulse is *not* orthogonal to its shift by multiples of T_s. But then, the value Y_j computed above will not depend only on X_j, but on other members of the sequence $\{X_i\}$, too. Thus, we have inter symbol interference. Furthermore, $\{Z_j\}_{j=-\infty}^{\infty}$ is not an i.i.d. sequence unless $\{\psi(t - jT)\}_{j=-\infty}^{\infty}$ is an orthogonal sequence.

Problem 3. *(Mixed Questions)*

1. We use the Fourier transform property, namely, multiplication in the time domain is equivalent to convolution in the frequency domain. The Fourier transform of $\frac{\sin(\pi t)}{\pi t}$ is a rectangular pulse with amplitude 1 in $f \in [-\frac{1}{2}, \frac{1}{2}]$. Hence, the Fourier transform of $(\frac{\sin(\pi t)}{\pi t})^2$ is the convolution of the rectangular pulse with itself which will be a triangular pulse with peak value 1 at $f = 0$ and with support $f \in [-1, 1]$. The Fourier transform of $\cos(2\pi t)$ is $\frac{1}{2}\delta(f-1) + \frac{1}{2}\delta(f+1)$. Hence the Fourier transform of $x(t)$ will be the convolution of the Fourier transform of $\cos(2\pi t)$ with the triangular pulse. A simple drawing of the frequency domain shows that the $x_{\mathcal{F}}(f)$ is two triangular pulses, both of them have peak value $\frac{1}{2}$. One of them has the support $f \in [-2, 0]$ and the other in $f \in [0, 2]$. As $x_{\mathcal{F}}$ is a base band signal with bandwidth 4 then the maximum sampling time possible to avoid aliasing is $\frac{1}{4}$.

2. It is easy to see that $\int_0^1 s_1(t)s_2(t)dt = 0$ hence s_1 and s_2 are orthogonal and the dimension of the signal set is at least two. Furthermore, we can write $s_3(t) = \sin^2(\pi t) = \frac{1-\cos(2\pi t)}{2} = \frac{1}{2}s_1(t) - \frac{1}{2}s_2(t)$. Hence the dimensionality of the signal set is two.

3. It is easy to check that $p(t)$ meets the Nyquist criterion. In other words, a simple drawing of the frequency domain shows that $\frac{1}{T}\sum |p_{\mathcal{F}}(f - \frac{n}{T})|^2 = 1$. Hence $p(t)$ is

orthogonal to its shifted versions by an integer multiple of T. In other words $\int p(t)p(t-nT)dt = 0, \quad n \neq 0$. Hence $\int p(t)p(t-3T)dt = 0$.

Problem 4. *(Power Spectrum: Manchester Pulse)*

1. $r(t)$ is a rectangular pulse in the time domain hence its Fourier transform is sinc. Specifically, $r_{\mathcal{F}}(f) = \frac{\text{sinc}(\frac{\pi f T_s}{2})}{\pi f \sqrt{T_s}}$.

2. We can write $\phi(t)$ as $r(t) \star (\delta(t - \frac{T_s}{4}) - \delta(t - \frac{3T_s}{4}))$. Hence using the fact that convolution in the time domain is equivalent to multiplication in the frequency domain we have:

$$\begin{aligned}
\phi_{\mathcal{F}}(f) &= (\exp(-j2\pi f \frac{T_s}{4}) - \exp(-j2\pi f \frac{3T_s}{4}))r_{\mathcal{F}}(f) \\
&= 2j \exp(-j\pi f T_s) \sin(\frac{\pi f T_s}{2}) \times \frac{\sin(\frac{\pi f T_s}{2})}{\sqrt{T_s}\pi f} \\
&= 2j \exp(-j\pi f T_s) \frac{\sin^2(\frac{\pi f T_s}{2})}{\sqrt{T_s}\pi f}
\end{aligned}$$

   Hence

$$|\phi_{\mathcal{F}}(f)|^2 = 4\frac{\sin^4(\frac{\pi f T_s}{2})}{T_s \pi^2 f^2}$$

3. $\{X_i\}$ are i.i.d random variables with mean zero. Hence $E\{X_i X_j\} = 0, \quad i \neq j$ and $E\{X_i^2\} = E_s$ so we can write $R_X[k] = E_s \delta[k]$ where $\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$. Hence in the summation $\sum R_X[k] e^{-j2\pi k f T_s}$ only the term corresponding to $k = 0$ remains which is equal to E_s. Putting together we have:

$$S_X(f) = E_s \frac{\sin^4(\frac{\pi f T_s}{2})}{(\frac{\pi f T_s}{2})^2}$$