ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
School of Computer and Communication Sciences

Principles of Digital Communications: Assignment date: Apr 18, 2012
Spring Semester 2012 Due date: Apr 25, 2012

Homework 8

Reading Part for this week: Chapter 4 until and including Section 4.4.2 (Growing BT Linearly with k).

Reading Part for next Wednesday: From Section 4.4.3 (Growing BT Exponentially with k) till Section 5.2 (The Ideal Lowpass Case). Read also Appendix 5A (Fourier Series) (10p)

Problem 1. *(Average Energy of PAM)*
Let U be a random variable uniformly distributed in $[-a, a]$ and let S be a discrete random variable, independent of U, which is uniformly distributed over $\{\pm a, \pm 3a, \cdots, \pm(m-1)a\}$ where m is an even integer. Let V be another random variable defined by $V \triangleq S + U$.

1. Find the distribution of the random variable V.
2. Find the variance of the random variables U and V.
3. Find the variance of S by using the variance of U and V. *(Hint: For independent random variables, the variance of the sum is the sum of the variances.)*
4. Notice that the variance of S is actually the average energy of a PAM constellation consisting of m points with nearest neighbor at distance $2a$. Verify your answer with the expression given in Example 4.4.57 of the lecture note.

Problem 2. *(Pulse Amplitude Modulated Signals)*
Consider using the signal set

$$s_i(t) = s_i\phi(t), \quad i = 0, 1, \ldots, m-1,$$

where $\phi(t)$ is a unit-energy waveform, $s_i \in \{\pm\frac{d}{2}, \pm\frac{3}{2}d, \ldots, \pm\frac{m-1}{2}d\}$, and $m \geq 2$ is an even integer.

1. Assuming that all signals are equally likely, determine the average energy $\mathcal{E}_s$ as a function of m. *(Hint: You may use the result of the previous problem.)*

2. Draw a block diagram for the ML receiver, assuming that the channel is AWGN with power spectral density $\frac{N_0}{2}$.

3. Give an expression for the error probability.

4. For large values of m, the probability of error is essentially independent of m but the energy is not. Let k be the number of bits you send every time you transmit $s_i(t)$ for some i, and rewrite $\mathcal{E}_s$ as a function of k. For large values of k, how does the energy behave when k increases by 1?

Problem 3. *(Root-Mean Square Bandwidth)*

The *root-mean square (rms) bandwidth* of a low-pass signal $g(t)$ of finite energy is defined by

$$W_{rms} = \left[\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 df}{\int_{-\infty}^{\infty} |G(f)|^2 df} \right]^{1/2}$$

where $|G(f)|^2$ is the energy spectral density of the signal. Correspondingly, the *root mean-square (rms) duration* of the signal is defined by

$$T_{rms} = \left[\frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt}{\int_{-\infty}^{\infty} |g(t)|^2 dt} \right]^{1/2}.$$

We want to show that, using the above definitions and assuming that $|g(t)| \to 0$ faster than $1/\sqrt{|t|}$ as $|t| \to \infty$, the time bandwidth product satisfies

$$T_{rms} W_{rms} \geq \frac{1}{4\pi}.$$

1. Use Schwarz inequality and the fact that for any $c \in \mathbb{C}$, $c + c^* = 2\Re\{c\} \leq 2|c|$, to prove that
$$\left\{ \int_{-\infty}^{\infty} [g_1^*(t) g_2(t) + g_1(t) g_2^*(t)] dt \right\}^2 \leq 4 \int_{-\infty}^{\infty} |g_1(t)|^2 dt \int_{-\infty}^{\infty} |g_2(t)|^2 dt.$$

2. In the above inequality insert
$$g_1(t) = t g(t)$$
and
$$g_2(t) = \frac{dg(t)}{dt}$$
and show that
$$\left[\int_{-\infty}^{\infty} t \frac{d}{dt} [g(t) g^*(t)] \, dt \right]^2 \leq 4 \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} \left| \frac{dg(t)}{dt} \right|^2 dt.$$

3. Integragte the left hand side by parts and use the fact that $|g(t)| \to 0$ faster than $1/\sqrt{|t|}$ as $|t| \to \infty$ to obtain

$$\left[\int_{-\infty}^{\infty} |g(t)|^2 dt\right]^2 \leq 4 \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} \left|\frac{dg(t)}{dt}\right|^2 dt.$$

4. Argue that the above is equivalent to

$$\int_{-\infty}^{\infty} |g(t)|^2 dt \int_{-\infty}^{\infty} |G(f)|^2 df \leq 4 \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} 4\pi^2 f^2 |G(f)|^2 df.$$

5. Complete the proof to obtain

$$T_{rms} W_{rms} \geq \frac{1}{4\pi}.$$

6. As a special case, consider a Gaussian pulse defined by

$$g(t) = \exp(-\pi t^2).$$

Show that for this signal

$$T_{rms} W_{rms} = \frac{1}{4\pi}$$

i.e., the above inequality holds with equality. *(Hint:* $\exp(-\pi t^2) \overset{\mathcal{F}}{\longleftrightarrow} \exp(-\pi f^2)$.*)*

Problem 4. *(Orthogonal Signal Sets)*
Consider the following situation: A signal set $\{s_j(t)\}_{j=0}^{m-1}$ has the property that all signals have the same energy $\mathcal{E}_s$ and that they are mutually orthogonal:

$$< s_i, s_j > \quad = \quad \mathcal{E}_s \delta_{ij}. \tag{1}$$

Assume also that all signals are equally likely. The goal is to translate this signal set into a minimum-energy signal set $\{s_j^*(t)\}_{j=0}^{m-1}$. It will prove useful to also introduce the unit-energy signals $\phi_j(t)$ such that $s_j(t) = \sqrt{\mathcal{E}_s}\phi_j(t)$.

1. Find the minimum-energy signal set $\{s_j^*(t)\}_{j=0}^{m-1}$.

2. What is the dimension of $span\{s_0^*(t), \ldots, s_{m-1}^*(t)\}$? For $m = 3$, sketch $\{s_j(t)\}_{j=0}^{m-1}$ and the corresponding minimum-energy signal set.

3. What is the average energy per symbol if $\{s_j^*(t)\}_{j=0}^{m-1}$ is used? What are the savings in energy (compared to when $\{s_j(t)\}_{j=0}^{m-1}$ is used) as a function of m?

Problem 5. *(m-ary Frequency Shift Keying)*
m-ary Frequency Shift Keying (MFSK) is a signaling method described as follows: $\boldsymbol{s}_i(t) = A\sqrt{\frac{2}{T}}\cos(2\pi(f_c + i\Delta f)t)\mathbf{1}_{[0,T]}(t), \quad i = 0, 1, \cdots, m-1$ where Δf is the minimum frequency separation of the different waveforms.

1. Assuming that $f_c T$ is an integer, find the minimum Δf as a function of T in order to have an orthogonal signal set.

2. In practice the signals $\boldsymbol{s}_i(t), \quad i = 0, 1, \cdots, m-1$ may be generated by changing the frequency of a signal oscillator. In passing from one frequency to another a phase shift θ is introduced. Again, assuming that $f_c T$ is an integer, determine the minimum Δf as a function of T in order to be sure that $A\cos(2\pi(f_c + i\Delta f)t + \theta_i)$ and $A\cos(2\pi(f_c + j\Delta f)t + \theta_j)$ are orthogonal for $i \neq j$ and for arbitrary θ_i and θ_j?

3. In practice we can't have complete control over f_c. In other words, it is not always possible to set $f_c T$ exactly to an integer number. Argue that if we choose $f_c >> m\Delta f$ then for all practical purposes the results of the two previous parts are still valid.

4. Assuming that all signals are equiprobable what is the mean power of the transmitter? How it behaves as a function of $k = \log_2(m)$?

5. What is the approximate frequency band of the signal set?

6. What is the BT product for this constellation? How it behaves as a function of k?

7. What is the main drawback of this signal set?