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Principles of Digital Communications: Summer Semester 2012
Assignment date: Mar 28, 2012
Due date: Apr 4, 2012

Solution of Homework 7

Problem 1. *(Non-coherent Detection)*

1. We simply find the conditional density of $r_0, ..., r_{N-1}$ under the two hypotheses and derive the likelihood ratio.
   For H_0, with pure noise assumption we have :

$$P(r_0, ..., r_{N-1}|H_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{\sum_{i=0}^{N-1} r_i^2}{2\sigma^2}\right)$$

And for H_1 we have :

$$\begin{aligned} P(r_0, ..., r_{N_1}|H_1) &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=0}^{N-1}(r_i - A\cos(2\pi f_0 i))^2\right) \\ &= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=0}^{N-1} r_i^2 + \frac{A}{\sigma^2}\sum_{i=0}^{N-1}\cos(2\pi f_0 i) r_i - \frac{NA^2}{2\sigma^2}\frac{1}{N}\sum_{i=0}^{N-1}\cos^2(2\pi f_0 i)\right) \\ &\approx \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=0}^{N-1} r_i^2 + \frac{A}{\sigma^2}\sum_{i=0}^{N-1}\cos(2\pi f_0 i) r_i - \frac{NA^2}{4\sigma^2}\right) \end{aligned}$$

Then we can compute the likelihood ratio :

$$L(r_0, ..., r_{N-1}) = \frac{P(r_0, ..., r_{N-1}|H_1)}{P(r_0, ..., r_{N-1}|H_0)} = \exp\left(\frac{A}{\sigma^2}\sum_{i=0}^{N-1} r_i \cos(2\pi f_0 i) - \frac{NA^2}{4\sigma^2}\right) \underset{H_1}{\overset{H_0}{\lessgtr}} \frac{P_0}{P_1} = 1,$$

which gives us the following MAP rule :

$$\sum_{i=0}^{N-1} r_i \cos(2\pi f_0 i) \underset{H_1}{\overset{H_0}{\lessgtr}} \frac{NA}{4}.$$

Now we can compute the conditional and mean error probabilities.

Under $H_0: \quad r_i \sim N(0, \sigma^2)$ which implies that

$$\sum_{i=0}^{N-1} r_i \cos(2\pi f_0 i) \sim N\left(o, \sigma^2 \sum_{i=0}^{N-1} \cos^2(2\pi f_0 i)\right) \approx N\left(0, \frac{N\sigma^2}{2}\right).$$

Hence we obtain

$$\begin{aligned} P(E|H_0) &= P\left(N\left(0, \frac{N\sigma^2}{2}\right) > \frac{NA}{4}\right) \\ &= Q\left(\frac{\frac{NA}{4}}{\sqrt{\frac{N\sigma^2}{2}}}\right) \\ &= Q\left(\sqrt{N\frac{A^2}{8\sigma^2}}\right) = Q\left(\sqrt{N\frac{SNR}{8}}\right). \end{aligned}$$

Similarly under $H_1 : r_i \sim N(A\cos(2\pi f_0 i), \sigma^2)$. If we define $Z \stackrel{def}{=} \sum_{i=0}^{N-1} r_i \cos(2\pi f_0 i)$ then Z is a Gaussian random variables and

$$\begin{aligned} E[Z] &= \sum A\cos^2(2\pi f_0 i) \approx \frac{NA}{2} \\ Var[Z] &= \sigma^2 \sum_{i=0}^{N-1} \cos^2(2\pi f_0 i) = \frac{N\sigma^2}{2}. \end{aligned}$$

Hence $Z \sim N\left(\frac{NA}{2}, \frac{N\sigma^2}{2}\right)$. For error probability under H_1 we have

$$P(E|H_1) = P\left(Z < \frac{NA}{4}\right) = Q\left(\frac{\frac{NA}{2} - \frac{NA}{4}}{\sqrt{\frac{N\sigma^2}{2}}}\right) = Q\left(\sqrt{N\frac{\text{SNR}}{8}}\right).$$

Combining the conditional error probabilities we obtain the mean error probability as

$$P(E) = \frac{1}{2}P(E|H_0) + \frac{1}{2}P(E|H_1) = Q\left(\sqrt{N\frac{\text{SNR}}{8}}\right)$$

2. You may know from signal processing courses that $\sum_{i=0}^{N-1} r_i \cos(2\pi f_0 i)$ is simply the Discrete Cosine Transform of the sequence $(r_0, r_1, ..., r_{N-1})$. Actually the MAP rule tries to analyze the signal in the frequency domain. If it observes any peak of considerable height at frequency f_0, it chooses H_1, otherwise it chooses H_0, which intuitively seems correct.

3. The structure of the MAP rule does not change. In other words, the MAP rule computes $\sum_{i=0}^{N-1} r_i \cos(2\pi f_0 i)$ and compares it with $\frac{NA}{4}$ to decide H_0 or H_1. Now because of the phase uncertainty the received signal we have $r_i = A\cos(2\pi f_0 i + \theta) + Z_i$.

Assuming that $Z = \sum_{i=0}^{N-1} r_i \cos(2\pi f_0 i)$ as before, which is a Gaussian random variable, we compute its mean and variance under H_1.

$$\begin{aligned} E[Z|H_1] &= A\sum_{i=0}^{N-1}\cos(2\pi f_0 i)\cos(2\pi f_0 i + \theta_0) \\ &= A\left(\sum_{i=0}^{N-1}\cos^2(2\pi f_0 i)\cos(\theta_0) - \sum_{i=0}^{N-1}\cos(2\pi f_0 i)\sin(2\pi f_0 i)\sin(\theta_0)\right) \\ &\approx \frac{NA}{2}\cos(\theta_0), \end{aligned}$$

where we used the trigonometric identity $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.

$$Var[Z] = \sigma^2\sum_{i=0}^{N-1}\cos^2(2\pi f_0 i) = \frac{N\sigma^2}{2},$$

which implies that :

$$Z|_{H_1} \sim N\left(\frac{NA}{2}\cos(\theta_0), \frac{N\sigma^2}{2}\right).$$

Hence we obtain the conditional error probability as

$$\begin{aligned} P(E|H_1) = P\left(Z < \frac{NA}{4}\right) \\ = Q\left(\frac{\frac{NA}{2}\cos(\theta_0) - \frac{NA}{4}}{\sqrt{\frac{N\sigma^2}{2}}}\right) \\ = Q\left((2\cos(\theta_0)-1)\sqrt{N\frac{\text{SNR}}{8}}\right) \end{aligned}$$

We see that if $|\theta_0| > \frac{\pi}{3}, \quad P(E|H_1) > \frac{1}{2}$.

4. , 5. The conditional error probability under H_0 does not depend on θ_0 because under H_0 the transmitter does not send any signal and we receive pure noise. Hence

$$P(E) = \frac{1}{2}\left\{Q\left(\sqrt{N\frac{\text{SNR}}{8}}\right) + Q\left((2\cos(\theta_0)-1)\sqrt{N\frac{\text{SNR}}{8}}\right)\right\}$$

We can simply see that in the case $\theta_0 = \frac{\pi}{2}$:

$$P(E) = \frac{1}{2}\left(Q\left(\sqrt{N\frac{\text{SNR}}{8}}\right) + Q\left(-\sqrt{N\frac{\text{SNR}}{8}}\right)\right) = \frac{1}{2},$$

where we used the identity $Q(x) + Q(-x) = 1, \quad x \in \mathbb{R}$, and even worse, when $\theta_0 = \pi$ (complete phase change),

$$P(E) = \frac{1}{2}\left(Q\left(\sqrt{N\frac{\text{SNR}}{8}}\right) + Q\left(-3\sqrt{N\frac{\text{SNR}}{8}}\right)\right) > \frac{1}{2},$$

This shows that the phase uncertainty can completely disrupt the communication.

Problem 2. *(Gram-Schmidt Procedure On Tuples)*
We denote inner product by $<,>$, norm by $||.||$ intermediate vectors by ϕ and final normalized vector by ψ. We start from β_1.

1. $||\beta_1|| = \sqrt{<\beta_1, \beta_1>} = \sqrt{3}$. $\psi_1 = \frac{\beta_1}{||\beta_1||} = (\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.
2. $<\psi_1, \beta_2> = \sqrt{3}$. $\phi_2 = \beta_2 - \sqrt{3}\psi_1 = (1, 1, -1, 0)$. $||\phi_2|| = \sqrt{3}$ and so $\psi_2 = \frac{\phi_2}{||\phi_2||} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0)$.
3. $<\psi_1, \beta_3> = 0$ and $<\psi_2, \beta_3> = 0$. $\phi_3 = \beta_3 - 0\psi_1 + 0\psi_2 = (1, 0, 1, -2)$ and $||\phi_3|| = \sqrt{1+1+4} = \sqrt{6}$ and so $\psi_3 = \frac{\phi_3}{||\phi_3||} = (\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})$.
4. $<\psi_1, \beta_4> = \sqrt{3}$, $<\psi_2, \beta_4> = 0$ and $<\psi_3, \beta_4> = \sqrt{6}$. $\phi_4 = \beta_4 - \sqrt{3}\psi_1 - 0\psi_2 - \sqrt{6}\psi_3 = (0, 0, 0, 0)$.

As can be seen the last vector is zero and this shows that the dimensionality of the space spanned by $\beta_1, \cdots, \beta_4$ is only 3 not 4. So the other benefit of Gram-Schmidt orthogonalization is that it gives us the dimension of the space spanned by initial vectors.

Problem 3. *(Gram-Schmidt Procedure On Waveforms)*

An orthonormal basis may be found using the so-called Gram-Schmidt procedure.

1. We use Gram-Schmidt procedure:
   (a) The first step is to normalize the function $s_1(t)$, i.e. the first function of the basis that we are looking for is

   $$\phi_0(t) \quad = \quad \frac{s_0(t)}{||s_0(t)||} = \frac{s_0(t)}{\sqrt{\int s_0(t)^2 dt}} = \frac{s_0(t)}{\sqrt{\int_0^1 4t^2 dt}} = \frac{\sqrt{3}}{2}s_0(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sqrt{3}t & \text{if } 0 < t < 1 \\ 0 & \text{if } t > 1 \end{cases}.$$

   (b) Next, we subtract from $s_1(t)$ the components that are in the span of the currently established part of the basis, i.e. in the span of $\{\phi_0(t)\}$. This can be achieved by projecting $s_1(t)$ onto $\phi_0(t)$ and then subtracting this projection from $s_1(t)$, i.e.

$$\begin{aligned}
e_1(t) &= s_1(t) - \langle s_1(t), \phi_0(t)\rangle \phi_0(t) = s_1(t) - \left(\int s_1(t)\phi_0(t)dt\right)\phi_0(t) \\
&= s_1(t) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{4}{3}\right)\phi_0(t) \\
&= s_1(t) - \frac{2}{\sqrt{3}}\phi_0(t) \\
&= s_1(t) - s_0(t)
\end{aligned}$$

From this, we find the second basis element as

$$\phi_1(t) = \frac{e_1(t)}{||e_1(t)||}. = \begin{cases} 0 & \text{if } t < 1 \\ \sqrt{3} - \sqrt{3}(t-1) & \text{if } 1 < t < 2 \\ 0 & \text{if } t > 2 \end{cases}$$

(c) Again, we subtract from $s_2(t)$ the components that are in the span of the currently established part of the basis, i.e. in the span of $\{\phi_0(t), \phi_1(t)\}$. This can be achieved by projecting $s_2(t)$ onto $\phi_0(t)$ and $\phi_1(t)$ and then subtracting both these projections from $s_2(t)$. For this step, it is *essential* that the basis elements $\{\phi_0(t), \phi_1(t)\}$ be orthonormal. Make sure you understand why. Continuing the derivation, we obtain

$$\begin{aligned}
e_2(t) &= s_2(t) - \langle s_2(t), \phi_0(t)\rangle\phi_0(t) - \langle s_1(t), \phi_1(t)\rangle\phi_1(t) \\
&= s_2(t) - \left(\int s_2(t)\phi_0(t)dt\right)\phi_0(t) - \left(\int s_2(t)\phi_1(t)dt\right)\phi_1(t) \\
&= s_2(t) - 0 - e_1(t) \\
&= s_2(t) - s_1(t) + s_0(t),
\end{aligned}$$

and from this, we find the third basis element as

$$\phi_2(t) = \frac{e_2(t)}{||e_2(t)||} = \begin{cases} 0 & \text{if } t < 2 \\ -\sqrt{3}(t-2) & \text{if } 2 < t < 3 \\ 0 & \text{if } t > 3 \end{cases}.$$

2. By definition we can write $V_1(t)$ and $V_2(t)$ as follows

$$V_1(t) = 3\phi_0(t) - \phi_1(t) + \phi_2(t) = \begin{cases} 3\sqrt{3}t & \text{if } 0 < t < 1 \\ -(\sqrt{3} - \sqrt{3}(t-1)) & \text{if } 1 < t < 2 \\ -\sqrt{3}(t-2) & \text{if } 2 < t < 3 \end{cases}$$

and

$$V_2(t) = -\phi_0(t) + 2\phi_1(t) + 3\phi_2(t) = \begin{cases} -\sqrt{3}t & \text{if } 0 < t < 1 \\ 2(\sqrt{3} - \sqrt{3}(t-1)) & \text{if } 1 < t < 2 \\ -3\sqrt{3}(t-2) & \text{if } 2 < t < 3 \end{cases}$$

3. We know that $V_1(t)$ and $V_2(t)$ are both real, thus

$$\begin{aligned}\langle V_1(t), V_2(t)\rangle &= \int V_1(t)V_2(t)dt \\ &= \langle V_1, V_2\rangle \\ &= -3*1-1*2+1*3 \\ &= -2\end{aligned}$$

Problem 4. *(Matched Filter Intuition)*

1. The Cauchy-Schwarz inequality states

$$|\langle x, y\rangle| \leq ||x||\ ||y||$$

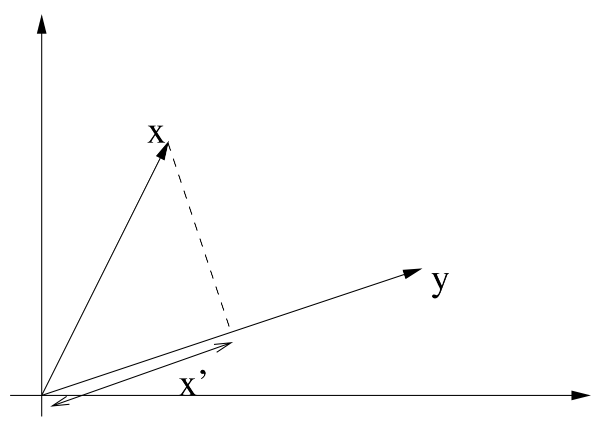
with equality if and only if $x = \alpha y$ for some scalar α. For our problem, we can write

$$|\langle s, \phi\rangle|^2 \leq |s|^2\ |\phi|^2 = |s|^2$$

with equality if and only if $\phi = \alpha s$ for some scalar α. Thus, the maximizing $\phi(t)$ is simply a scaled version of $s(t)$.

Note: In two dimensions, we have $|\langle x, y\rangle| \leq ||x||\ ||y||\ \cos\alpha$, where α is the angle between the two vectors; then, it is clear that the maximum is achieved when $\cos\alpha = 1$ $\Leftrightarrow \alpha = 0$ (or $\alpha = k2\pi$). Thus, x and y are colinear.

2. The inner product $\langle x, y\rangle$ is (using the definitions in the figure below) just the product of the length x' and the length y, i.e. $\langle x, y\rangle = ||x'||\ ||y||$. But it is immediately clear that $||x'||$ is maximal when x points in the same direction as y.



3. Denote $s = (s_1, s_2)$ and $\phi = (\phi_1, \phi_2)$. The problem is

$$\max_{\phi_1,\phi_2} (s_1\phi_1 + s_2\phi_2) \text{ subject to } \phi_1^2 + \phi_2^2 = 1.$$

Thus, we can reduce by setting $\phi_2 = \sqrt{1-\phi_1^2}$ to obtain

$$\max_{\phi_1} \left(s_1\phi_1 + s_2\sqrt{1-\phi_1^2} \right).$$

This maximum is found by taking the derivative:

$$\frac{d}{d\phi_1}\left(s_1\phi_1 + s_2\sqrt{1-\phi_1^2} \right) \quad = \quad s_1 - s_2\frac{\phi_1}{\sqrt{1-\phi_1^2}}.$$

Setting this equal to zero yields $s_1 = s_2\frac{\phi_1}{\sqrt{1-\phi_1^2}}$, i.e.

$$s_1^2 = s_2^2\frac{\phi_1^2}{1-\phi_1^2}.$$

This immediately gives $\phi_1 = \frac{s_1}{\sqrt{s_1^2+s_2^2}}$ and thus $\phi_2 = \frac{s_2}{\sqrt{s_1^2+s_2^2}}$, as expected.

Note: the goal of this exercise was to display yet another way to derive the matched filter.

4. Passing an input $s(t)$ through a filter with impulse response $h(t)$ generates output waveform $y(t) = \int s(\tau)h(t-\tau)d\tau$. If this waveform $y(t)$ is sampled at time $t = T$, then the output sample is:

$$y(T) = \int s(\tau)h(T-\tau)d\tau \tag{1}$$

An example signal $s(\tau)$ is shown in Figure 1(a). The filter is then the waveform shown in 1(b), and the convolution term of the filter in 1(c). Finally, the filter term $h(T-\tau)$ of Equation 1 is shown in 1(d). One can see that $h(T-\tau) = s(\tau)$, so indeed

$$y(T) = \int s(\tau)h(T-\tau)d\tau = \int s^2(\tau)d\tau = \int_0^T s^2(\tau)d\tau.$$

(v) Denote the signal spectrum by

$$S(f) \quad = \quad \int s(t)e^{-j2\pi ft}dt = |S(f)|e^{j\theta(f)}.$$

Then, the spectrum of the matched filter can be written as

$$\begin{aligned} H(f) \quad &= \quad \int h(t)e^{-j2\pi ft}dt = \int s(T-t)e^{-j2\pi ft}dt \\ &= \quad e^{-j2\pi fT}S^*(f) = e^{-j(\theta(f)+2\pi fT)}|S(f)|, \end{aligned}$$

where $S^*(f)$ is the complex conjugate. Now consider the signal $f(t)$ at the output of the matched filter. It is the convolution of the signal $s(t)$ with the matched filter

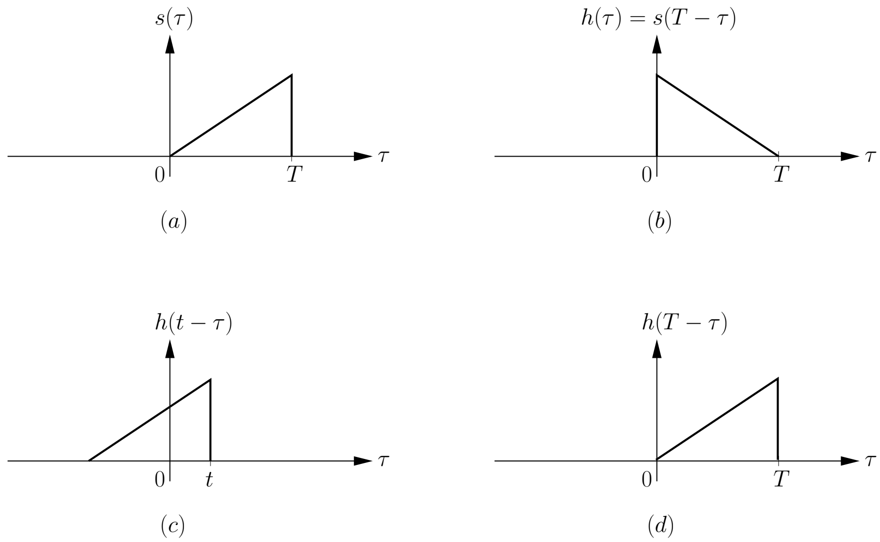


Figure 1: Signal and the impulse response waveforms

impulse response $h(t)$. As an inverse Fourier transform,

$$\begin{aligned} f(t) &= \int S(f)H(f)e^{j2\pi ft}df \\ &= \int |S(f)|^2 e^{-j2\pi fT} e^{j2\pi ft}df \\ &= \int |S(f)|^2 e^{j2\pi f(t-T)}df. \end{aligned}$$

Obviously, if $t = T$, all components in the integral "add in phase", and we simply get

$$f(t = T) = \int |S(f)|^2 df.$$

Problem 5. *(AWGN Channel And Sufficient Statistic)*

For the first part we have:

1. Under hypothesis $H = i$, the received waveform is $\boldsymbol{Y}(t) = \boldsymbol{s}_i(t) + \boldsymbol{Z}(t)$ and there is one-to-one correspondence between $\boldsymbol{Y}(t)$ and $\boldsymbol{Y} = (Y_0, Y_1, Y_2)^T$ where $Y_i = < \boldsymbol{Y}(t), \phi_i(t) >$. Hence, $\boldsymbol{Y}$ is a sufficient statistic. It is straight forward to verify that when $H = i$, $\boldsymbol{Y} = \boldsymbol{s}_i + \boldsymbol{N}$.

2. The third component of $\boldsymbol{s}_i$ is zero for all i. Further more N_0, N_1 and N_2 are zero mean iid Gaussian random variables. Hence, $f_{\boldsymbol{Y}|H}(\boldsymbol{y}|i) = f_{N_0}(y_0 - s_{i0}) f_{N_1}(y_1 - s_{i1}) f_{N_2}(y_2)$ which is in the form $g_i(T(\boldsymbol{y}))h(\boldsymbol{y})$ for $T(\boldsymbol{y}) = (y_0, y_1)^T$ and $h(\boldsymbol{y}) = f_{N_2}(y_2)$. Hence, by the Fisher-Neyman factorization theorem, $T(\boldsymbol{Y}) = (Y_0, Y_1)^T$ is a sufficient statistic.

For the second part of the problem in which $N_2 = N_1$ we have:

1. If we have only (Y_0, Y_1) then hypothesis testing problem will be $H = i \ : \ (Y_0, Y_1) = (s_{i0}, s_{i1}) + (N_0, N_1) \quad i = 0, 1$. Using the fact that $\boldsymbol{s}_0 = (1, 0, 0)^T$ and $\boldsymbol{s}_1 = (0, 1, 0)^T$, the ML test becomes $y_0 - y_1 \underset{H_1}{\overset{H_0}{\gtrless}} 0$. Under $H = 0$, $Y_0 - Y_1$ is a Gaussian random variable with mean 1 and variance $2\sigma^2$ and so $P_e(0) = Q(\frac{1}{\sqrt{2}\sigma})$. By symmetry $P_e(1) = Q(\frac{1}{\sqrt{2}\sigma})$ and so the probability of the error will be $P_e = \frac{1}{2}(P_e(0) + P_e(1)) = Q(\frac{1}{\sqrt{2}\sigma})$.

2. Now assume that we have access to Y_0, Y_1 and Y_2. Y_2 contains N_2 under both hypotheses. Hence, $Y_1 - Y_2 = s_{i1} + N_1 - N_2 = s_{i1}$. This shows that at the receiver we can observe the second component of $\boldsymbol{s}_i$ without noise. As the second component is different under both hypotheses, we can make an error-free decision about H and the decision rule will be:
$$\hat{H} = \begin{cases} 0 & y_1 - y_2 = 0 \\ 1 & y_1 - y_2 = 1 \end{cases}$$
Clearly this decision rule minimizes the probability of the error.

3. Y_2 allows us to reduce the probability of the error. Hence, (Y_0, Y_1) can't be a sufficient statistic.