Problem 1. Note that \( E_0 = E_1 \cup E_2 \cup E_3 \).

(a) (1) For disjoint events, \( P(E_0) = P(E_1) + P(E_2) + P(E_3) \), so \( P(E_0) = 3/4 \).

(2) For independent events, \( 1 - P(E_0) \) is the probability that none of the events occur, which is the product of the probabilities that each one doesn’t occur. Thus \( 1 - P(E_0) = (3/4)^3 \) and \( P(E_0) = 37/64 \).

(3) If \( E_1 = E_2 = E_3 \), then \( E_0 = E_1 \) and \( P(E_0) = 1/4 \).

(b) (1) From the Venn diagram in Fig. 1, \( P(E_0) \) is clearly maximized when the events are disjoint, so \( \max P(E_0) = 3/4 \).

\[
\begin{align*}
E_1 & \quad E_2 \\
& \quad \quad E_3
\end{align*}
\]

Figure 1: Venn Diagram for problem 1 (b)(1)

(2) The intersection of each pair of sets has probability 1/16. As seen in Fig. 2, \( P(E_0) \) is maximized if all these pairwise intersections are identical, in which case \( P(E_0) = 3(1/4 - 1/16) + 1/16 = 5/8 \). One can also use the formula \( P(E_0) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) \), and notice that all the terms except the last is fixed by the problem, and the last term cannot be made more than \( \min_{i,j} P(E_i \cap E_j) = 1/16 \).

\[
\begin{align*}
E_1 & \quad E_2 \\
& \quad \quad E_3
\end{align*}
\]

Figure 2: Venn Diagram for problem 1 (b)(2)

Problem 2. Let \( L \) be the event that the loaded die is picked and \( H \) the event that the honest die is picked. Let \( A_i \) be the event that \( i \) is turned up on the first roll, and \( B_i \) be the event that \( i \) is turned up on the second roll. We are given that \( P(L) = 1/3, P(H) = 2/3 \); \( P(A_i \mid L) = 2/3; P(A_i \mid L) = 1/15 \) \( 2 \leq i \leq 6; P(A_i \mid H) = 1/6 \) \( 1 \leq i \leq 6 \). Then
\[
P(L \mid A_1) = \frac{P(L, A_1)}{P(A_1)} = \frac{P(A_1 \mid L) P(L)}{P(A_1 \mid L) P(L) + P(A_1 \mid H) P(H)} = \frac{2}{3}.
\]
This is the probability that the loaded die was picked conditional on the first roll showing a 1. For two rolls we make the assumption from the physical mechanism involved in rolling a die that the outcome on the two successive rolls of a given die are independent. Thus \( P(A_1B_1 \mid L) = (2/3)^2 \) and \( P(A_1B_1 \mid H) = (1/6)^2 \). It follows as before that 
\[
P(L \mid A_1B_1) = \frac{8}{9}.
\]

**Problem 3.** Since \( A, B, C, D \) form a Markov chain their probability distribution is given by

\[
p(a)p(b|a)p(c|b)p(d|c)
\]

(a) Yes: Summing (1) over \( d \) shows that \( A, B, C \) have the probability distribution \( p(a)p(b|a)p(c|b) \).

(b) Yes: The reverse of a Markov chain is also a Markov chain. Applying this to \( A, B, C, D \) and using part (a) we get that \( D, C, B \) is a Markov chain. Reversing again we get the desired result.

(c) Yes: Since \( A, B, C, D \) is a Markov chain, given \( C, D \) is independent of \( B \), and thus \( p(d|c) = p(d|(b,c)) \). So (1) can be written as

\[
p(a, (b, c), d) = p(a)p((b, c)|a)p(d|(b, c))
\]

(d) Yes, by a similar (in fact easier) reasoning as (c).

**Problem 4.** No. Take for example \( A = D \) and let \( A \) be independent of the pair \( (B, C) \). Then both \( A, B, C \) and \( B, C, A \) (same as \( B, C, D \)) are Markov chains. But \( A, B, C, D \) is not: \( A \) is not independent of \( D \) when \( B \) and \( C \) are given.

**Problem 5.**

(a) Note that the event \( N = n \) is the same as the coin falling tails \( n - 1 \) times followed by it falling heads. Since the coin flips are independent and they are fair, we get \( \Pr(N = n) = 2^{-(n-1)}2^{-1} = 2^{-n} \). Using Bayes’ rule:

\[
\Pr(N = n \mid N \in \{n, n+1\}) = \frac{\Pr(N = n)}{\Pr(N \in \{n, n+1\})} = \frac{2^{-n}}{2^{-n} + 2^{-(n+1)}} = \frac{2}{3}
\]

(b) The only way we find 1 franc in the chosen box is when \( N = 1 \) and we have chosen the box with the smaller amount of money. The other box thus contains 3 francs.

(c) If we find \( 3^n \) francs in the chosen box, we know that \( N \) is either \( n \) (and the other box contains \( 3^{n-1} \) francs) or \( n + 1 \) (and the other box contains \( 3^{n+1} \) francs). Using part (a), \( N = n \) with probabity \( 2/3 \), and \( N = n + 1 \) with probability \( 1/3 \). Thus the expected money in the other box is

\[
\frac{2}{3}3^{n-1} + \frac{1}{3}3^{n+1} = \frac{11}{9}3^n
\]

francs.

(d) Indeed, no matter what we find in the chosen box, the expected amount in the other box is more then the amount found in the chosen box (3 vs 1 as in part (b) or 11/9 times as in part (c)). We thus have, with \( X \) and \( Y \) representing the amount in the two boxes,

\[
E[X|Y] > Y \quad \text{and} \quad E[Y|X] > X.
\]
This appears to be a paradox if we take expectations again to obtain

\[ E[X] > E[Y] \quad \text{and} \quad E[Y] > E[X]. \]

However, some thought reveals that \( E[X] \) and \( E[Y] \) do not exist, and so the last equation is without content: Since \( \Pr(N = n) = 2^{-n} \), the expected amount of money in the box with the smaller amount is \( \sum_{n \geq 1} 2^{-n}3^{n-1} \) which is a divergent series.

**Problem 6.**

(a) 

\[
E[X + Y] = \sum_{x,y} (x + y)P_{XY}(x, y) \\
= \sum_{x,y} xP_{XY}(x, y) + \sum_{x,y} yP_{XY}(x, y) \\
= \sum_{x} xP_X(x) + \sum_{y} yP_Y(y) \\
= E[X] + E[Y].
\]

Note that independence is not necessary here and that the argument extends to non-discrete variables if the expectation exists.

(b) 

\[
E[XY] = \sum_{x,y} xyP_{XY}(x, y) \\
= \sum_{x,y} xP_X(x)P_Y(y) \\
= \sum_{x} xP_X(x) \sum_{y} yP_Y(y) \\
= E[X]E[Y].
\]

Note that the statistical independence was used on the second line. Let \( X \) and \( Y \) take on only the values \( \pm 1 \) and 0. An example of uncorrelated but dependent variables is

\[ P_{XY}(1, 0) = P_{XY}(0, 1) = P_{XY}(-1, 0) = P_{XY}(0, -1) = \frac{1}{4}. \]

An example of correlated and dependent variables is

\[ P_{XY}(1, 1) = P_{XY}(-1, -1) = \frac{1}{2}. \]

(c) Using (a), we have

\[
\sigma_{X+Y}^2 = E[(X - E[X] + Y - E[Y])^2] \\
= E[(X - E[X])^2] + 2E[(X - E[X])(Y - E[Y])] + E[(Y - E[Y])^2].
\]

The middle term, from (a), is \( 2(E[XY] - E[X]E[Y]) \). For uncorrelated variables that is zero, leaving us with \( \sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2. \)
Problem 7. We solve the problem for a general vehicle with $n$ wheels.

(a) Out of $n!$ possible orderings $(n-1)!$ has the tyre 1 in its original place. Thus tyre 1 is installed in its original position with probability $1/n$.

(b) All tyres end up in their original position in only 1 of the $n!$ orders. Thus the probability of this event is $1/n!$.

(c) Let $X_i$ be the indicator random variable that tyre $i$ is installed in its original position, so that the number of tyres installed in their original positions is $N = \sum_{i=1}^{n} X_i$. By (a), $E[X_i] = 1/n$. By the linearity of expectation, $E[N] = n(1/n) = 1$. Note that the linearity of the expectation holds even if the $X_i$’s are not independent (as it is in this case).

(e) Let $A_i$ be the event that the $i$th tyre remains in its original position. Then, the event we are interested in is the complement of the event $\bigcup_i A_i$ and thus has probability $1 - \Pr(\bigcup_i A_i)$. Furthermore, by the inclusion/exclusion formula,

$$\Pr(\bigcup_i A_i) = \sum_i \Pr(A_i) - \sum_{i_1 < i_2} \Pr(A_{i_1} \cap A_{i_2}) + \sum_{i_1 < i_2 < i_3} \Pr(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \ldots$$

The $j$th sum above consists of $\binom{n}{j}$ terms, each term having the value $\Pr(A_1 \cap \cdots \cap A_j)$. Note that this is the probability of the event that tyres 1 through $j$ have remained in their original positions, and equals $(n - j)!/n!$. Consequently,

$$\Pr(\bigcup_i A_i) = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{(n-j)!}{n!} = \sum_{j=1}^{n} (-1)^{j-1} 1/j!,$$

and the event that no tyre remains in its original position has probability

$$1 - \Pr(\bigcup_i A_i) = \sum_{j=0}^{n} \frac{(-1)^j}{j!}.$$

(For the case $n = 4$, the value is $3/8$.)