ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 14 Information Theory and Coding
Homework 7 November 1, 2011

PROBLEM 1. Let the alphabet be $\mathcal{X} = \{a, b\}$. Consider the infinite sequence $X_1^\infty =$ *abababababababab*......

(a) What is the compressibility of $\rho(X_1^\infty)$ using finite-state machines (FSM) as defined in class? Justify your answer.

(b) Design a specific FSM, call it M, with at most 4 states and as low a $\rho_{\mathrm{M}}(X_1^\infty)$ as possible. What compressibility do you get?

(c) Using only the result in point (a) but no specific calculations, what is the compressibility of X_1^∞ under the Lempel-Ziv algorithm, i.e., what is $\rho_{\mathrm{LZ}}(X_1^\infty)$?

(d) Re-derive your result from point (c) but this time by means of an explicit computation.

PROBLEM 2.

(a) Show that $I(U; V) \geq I(U; V|T)$ if T, U, V form a Markov chain, i.e., conditional on U, the random variables T and V are independent.

Fix a conditional probability distribution $p(y|x)$, and suppose $p_1(x)$ and $p_2(x)$ are two probability distributions on $\mathcal{X}$.

For $k \in \{1, 2\}$, let I_k denote the mutual information between X and Y when the distribution of X is $p_k(\cdot)$.

For $0 \leq \lambda \leq 1$, let W be a random variable, taking values in $\{1, 2\}$, with

$$\Pr(W = 1) = \lambda, \quad \Pr(W = 2) = 1 - \lambda.$$

Define

$$p_{W,X,Y}(w, x, y) = \begin{cases} \lambda p_1(x)p(y|x) & \text{if } w = 1 \\ (1 - \lambda)p_2(x)p(y|x) & \text{if } w = 2. \end{cases}$$

(b) Express $I(X; Y|W)$ in terms of I_1, I_2 and λ.

(c) Express $p(x)$ in terms of $p_1(x)$, $p_2(x)$ and λ.

(d) Using (a), (b) and (c) show that, for every fixed conditional distribution $p_{Y|X}$, the mutual information $I(X; Y)$ is a concave $\cap$ function of p_X.

PROBLEM 3. A source produces independent, equally probable symbols from an alphabet (a_1, a_2) at a rate of one symbol every 3 seconds. These symbols are transmitted over a binary symmetric channel which is used once each second by encoding the source symbol a_1 as 000 and the source symbol a_2 as 111. If in the corresponding 3 second interval of the channel output, any of the sequences 000,001,010,100 is received, a_1 is decoded; otherwise, a_2 is decoded. Let $\epsilon < 1/2$ be the channel crossover probability.

(a) For each possible received 3-bit sequence in the interval corresponding to a given source letter, find the probability that a_1 came out of the source given that received sequence.

(b) Using part (a), show that the above decoding rule minimizes the probability of an incorrect decision.

(c) Find the probability of an incorrect decision (using part (a) is not the easy way here).

(d) If the source is slowed down to produce one letter every $2n+1$ seconds, a_1 being encoded by $2n+1$ 0's and a_2 being encoded by $2n+1$ 1's. What decision rule minimizes the probability of error at the decoder? Find the probability of error as $n \to \infty$.

Problem 4. One is given a communication channel with transition probabilities $p(y|x)$ and channel capacity $C = \max_{P_X} I(X;Y)$. A helpful statistician preprocesses the output by forming $\tilde{Y} = g(Y)$. He claims that this will strictly improve the capacity.

(a) Show that he is wrong.

(b) Under what conditions does he not strictly decrease the capacity?

Problem 5. The Z-channel has binary input and output alphabets and transition probabilities $p(y|x)$ given by

$$p(0|0) = 1 \quad \text{and} \quad p(0|1) = \varepsilon.$$

Find the capacity of the Z-channel and the maximizing input probability distribution in terms of ε.