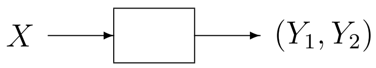
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 24 Information Theory and Coding

Homework 11 December 13, 2011

PROBLEM 1.



Consider the ordinary Gaussian channel with two correlated looks at X, i.e., $Y = (Y_1, Y_2)$, where

$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= X + Z_2 \end{aligned}$$

with a power constraint P on X, and (Z_1, Z_2) a Gaussian zero mean random vector with covariance K, where

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}.$$

Find the capacity C for

(a) $\rho = 1$.

(b) $\rho = 0$.

(c) $\rho = -1$.

PROBLEM 2. Consider a pair of parallel Gaussian channels, i.e.,

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

where

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right),$$

and there is a power constraint $E(X_1^2 + X_2^2) \leq P$. Assume that $\sigma_1^2 > \sigma_2^2$.

(a) Suppose we use the capacity achieving distribution as input. At what power does the channel stop behaving like a single channel with noise variance σ_2^2, and begin behaving like a pair of channels?

(b) Let $C_1(P)$ be the capacity of the channels described above (when the input is constrained to have a power not exceeding P). Let $C_2(P) = I(X_1, X_2; Y_1, Y_2)$ when both X_1 and X_2 are independent Gaussian random variables with variance equal to $\frac{P}{2}$. Show that $C_1(P) - C_2(P)$ tends to zero as P tends to infinity.

PROBLEM 3. Consider a vector Gaussian channel described as follows:

$$Y_1 = x + Z_1$$
$$Y_2 = Z_2$$

where x is the input to the channel constrained in power to P; Z_1 and Z_2 are jointly Gaussian random variables with $E[Z_1] = E[Z_2] = 0$, $E[Z_1^2] = E[Z_2^2] = \sigma^2$ and $E[Z_1 Z_2] = \rho\sigma^2$, with $\rho \in [-1, 1]$, and independent of the channel input.

(a) Consider a receiver that discards Y_2 and decodes the message based only on Y_1. What rates are achievable with such a receiver?

(b) Consider a receiver that forms $Y = Y_1 - \rho Y_2$, and decodes the message based only on Y. What rates are achievable with such a receiver?

(c) Find the capacity of the channel and compare it to the part (b).

PROBLEM 4. Consider an additive noise channel with input $x \in \mathbb{R}$, and output

$$Y = x + Z$$

where Z is a real random variable independent of the input x, has zero mean and variance equal to σ^2.

In this problem we prove in two different ways that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance. Let $\mathcal{N}_{\sigma^2}$ denote the Gaussian density with zero mean and variance σ^2.

First Method: Let X be a Gaussian random variable with zero-mean and variance P. Let $\mathcal{N}_P$ denote its density $\mathcal{N}_P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$.

(a) Show that

$$I(X;Y) = h(X) - h(X - \alpha Y|Y)$$

for any $\alpha \in \mathbb{R}$.

(b) Observe that

$$h(X - \alpha Y) \leq \frac{1}{2} \log 2\pi e E((X - \alpha Y)^2)$$

for any $\alpha \in \mathbb{R}$.

(c) Deduce from *(a)* and *(b)* that

$$I(X;Y) \geq h(X) - \frac{1}{2} \log 2\pi e E((X - \alpha Y)^2)$$

for any $\alpha \in \mathbb{R}$.

(d) Show that

$$E((X - \alpha Y)^2) \geq \frac{\sigma^2 P}{\sigma^2 + P}$$

with equality if and only if $\alpha = \frac{P}{P+\sigma^2}$.

(e) Deduce from (c) and (d) that

$$I(X;Y) \geq \frac{1}{2} \log \left(1 + \frac{P}{\sigma^2}\right)$$

and conclude that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance.

Second Method:

(a) Denote the input probability density by p_X. Verify that

$$I(X;Y) = \iint p_X(x) p_Z(y-x) \ln \frac{p_Z(y-x)}{p_Y(y)}\, dxdy \quad \text{nats.}$$

where p_Y is the probability density of the output when the input has density p_X.

(b) Now set $p_X = \mathcal{N}_P$. Verify that

$$\tfrac{1}{2} \ln\bigl(1 + P/\sigma^2\bigr) = \iint p_X(x) p_Z(y-x) \ln \frac{\mathcal{N}_{\sigma^2}(y-x)}{\mathcal{N}_{P+\sigma^2}(y)}\, dxdy.$$

(c) Still with $p_X = \mathcal{N}_P$, show that

$$\tfrac{1}{2} \ln(1 + P/\sigma^2) - I(X;Y) \leq 0.$$

[Hint: use (a) and (b) and $\ln t \leq t - 1$.]

(d) Show that an additive noise channel with noise variance σ^2 and input power P has capacity at least $\frac{1}{2}\log_2(1 + P/\sigma^2)$ bits per channel use. Conclude that the Gaussian channel has the smallest capacity among all additive noise channels of a given noise variance.