ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 16 Information Theory and Coding

Homework 10 December 6, 2011

PROBLEM 1. From symmetry, the capacity achieving distribution has to be $p(1) = p(-1) = \alpha$, and $p(0) = 1 - 2\alpha$ for some α. The cost constraint translates to

$$p(1) + p(-1) = 2\alpha \leq \beta.$$

Computing $I(X;Y)$ we get that

$$I(X;Y) = 2\alpha.$$

So, we want that α is the largest possible:

$$\alpha = \begin{cases} \beta/2, & \text{if } \beta \leq 1 \\ 1/2, & \text{if } \beta > 1 \end{cases}.$$

Hence, the capacity is

$$C = \begin{cases} \beta, & \text{if } \beta \leq 1 \\ 1, & \text{if } \beta > 1 \end{cases}.$$

PROBLEM 2.

(a) We know that log is concave, the sum of concave functions is concave again.

(b) The Kuhn-Tucker conditions are

$$\frac{\frac{1}{\sigma_i^2}}{1 + \frac{p_i}{\sigma_i^2}} = \mu, \text{ for } p_i > 0,$$
$$\frac{\frac{1}{\sigma_i^2}}{1 + \frac{p_i}{\sigma_i^2}} \leq \mu, \text{ for } p_i = 0,$$

With rearranging the terms we can get the desired form with $\lambda = \frac{1}{\mu}$.

(c) It directly follows from part (b).

(d) Water-filling: start with $\lambda = 0$ and start increasing λ until $\sum_i p_i = 1$.

PROBLEM 3. Since X and Z are both in the interval $[-1, 1]$, their sum $X + Z$ lies in the interval $[-2, +2]$. If we could *choose* the distribution of $X + Z$ as we wished (without the constraint that it has to be the sum of two independent random variables, one of which is uniform) we would have chosen it to be uniform on the interval $[-2, +2]$ to have the largest entropy. Observe now that if we choose X as the random variable that equals $+1$ with probability $1/2$ and -1 with probability $1/2$, then $X + Z$ is uniform in $[-2, +2]$ and thus this distribution maximizes the entropy. An alternate derivation is as follows: note that since X and Z are independent, the moment generating functions of the random variables involved satisfy $E[e^{s(X+Z)}] = E[e^{sX}]E[e^{sX}]$. Now, we know that $E[e^{sZ}] = \int e^{sz} f_Z(z)\, dz = \int_{-1}^{+1} \frac{1}{2} e^{sz}\, dz = [e^s - e^{-s}]/(2s)$. Similarly, if we want $X + Z$ to be uniform on $[-2, 2]$, we can compute $E[e^{s(X+Z)}] = [e^{2s} - e^{-2s}]/(4s)$. This then requires $E[e^{sX}] = \frac{1}{2}[e^{2s} - e^{-2s}]/[e^s - e^{-s}] =$

$\frac{1}{2}[e^s + e^{-s}]$ which is the moment generating function of a random variable which takes on the values $+1$ and -1, each with probability $1/2$.

Similarly, under the constraint XZ lies in the interval $[-1, +1]$, and the best we could hope is that XZ is uniform on this interval. But this can be achieved by making sure that X only takes on the values $+1$ or -1.

Problem 4.

(a)

$$-\int_0^\infty \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \log\left(\frac{1}{\lambda} e^{-\frac{x}{\lambda}}\right) \, \mathrm{d}x = \frac{1}{\lambda}\int_0^\infty x\frac{1}{\lambda} e^{-\frac{x}{\lambda}} \, \mathrm{d}x - \log\frac{1}{\lambda}\int_0^\infty \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \, \mathrm{d}x = 1 + \log\lambda$$

(b) Taking the hint:

$$\begin{aligned} 0 &\le D(q\|p) \\ &= \int q(x)\log\frac{q(x)}{p(x)} \, \mathrm{d}x \\ &= \int q(x)\log q(x) \, \mathrm{d}x + \int q(x)\log\frac{1}{p(x)} \, \mathrm{d}x \\ &= -h(q) + \int q(x)\log\frac{1}{p(x)} \, \mathrm{d}x. \end{aligned}$$

Now, note that $\log[1/p(x)]$ is of the form $\alpha + \beta x$, and since densities p and q have the same mean, we conclude that

$$\int q(x)\log\frac{1}{p(x)} \, \mathrm{d}x = \int p(x)\log\frac{1}{p(x)} \, \mathrm{d}x = h(p).$$

Thus, $0 \le -h(q) + h(p)$, yielding the desired conclusion.

Problem 5.

$$\begin{aligned} h(X) &= \frac{1}{2}\log(2\pi e\sigma_x^2) \\ h(Y) &= \frac{1}{2}\log(2\pi e\sigma_y^2) \\ h(X,Y) &= \frac{1}{2}\log((2\pi e)^2\det(K)) = \frac{1}{2}\log((2\pi e)^2(\sigma_x^2\sigma_y^2 - \rho^2\sigma_x^2\sigma_y^2) \\ I(X,Y) &= h(X) + h(Y) - h(X,Y) = \frac{1}{2}\log\frac{1}{1-\rho^2} \end{aligned}$$

Note that the result does not depend on σ_x, σ_y, which says that normalization does not change the mutual information.