ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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**Handout 16** Information Theory and Coding
Homework 10 December 6, 2011

PROBLEM 1. A discrete memoryless channel has three input symbols: $\{-1; 0; -1\}$, and two output symbols: $\{1; -1\}$. The transition probabilities are

$$p(-1|-1) = p(1|1) = 1, \quad p(1|0) = p(-1|0) = 0.5.$$

Find the capacity of this channel with cost constraint $\beta$, if the cost function is $b(x) = x^2$.

PROBLEM 2. For given positive numbers $\sigma_1^2, \dots, \sigma_K^2$ define the function

$$f(p_1, \dots, p_K) = \sum_{i=1}^{K} \log(1 + p_i/\sigma_i^2)$$

on the simplex $\{(p_1, \dots, p_K) : p_i \geq 0, \sum_i p_i = 1\}$.

(a) Show that $f$ is concave.

(b) Write the Kuhn-Tucker conditions for the $p$ that maximizes $f(p)$; show that they can be equivalently written as "there exists $\lambda$, such that

$$\begin{aligned} p_i &= \lambda - \sigma_i^2, \quad \text{for } i \text{ for which } p_i > 0 \\ 0 &\geq \lambda - \sigma_i^2, \quad \text{for } i \text{ for which } p_i = 0\text{''}. \end{aligned}$$

(c) Show that the maximizing $p$ can be written in the form

$$p_i = (\lambda - \sigma_i^2)^+$$

for some $\lambda$ where $a^+ = \max\{0, a\}$.

(d) Given what we have shown so far, the maximization of $f$ is reduced to finding the right $\lambda$. Describe a procedure to find this $\lambda$.

PROBLEM 3. Suppose $Z$ is uniformly distributed on $[-1, 1]$, and $X$ is a random variable, independent of $Z$, constrainted to take values in $[-1, 1]$. What distribution for $X$ maximizes the entropy of $X + Z$? What distribution of $X$ maximizes the entropy of $XZ$?

PROBLEM 4. Consider the exponential distribution with mean $\lambda$.

(a) Compute the differential entropy of the density function $p(x) = \frac{1}{\lambda} e^{-x/\lambda}$

(b) Show that among all non-negative random variables with mean $\lambda$ the exponential random variable has the largest differential entropy. Hint: let $p(x) = e^{-x/\lambda}/\lambda$ be the density of the exponential random variable and let $q(x)$ be some other density with mean $\lambda$. Consider $D(q\|p)$ and mimic the proof in the class for the maximal entropy of the Gaussian.

PROBLEM 5. Random variables $X$ and $Y$ are correlated Gaussian variables:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}_2 \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}; K = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} \right).$$

Find $I(X; Y)$.