ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
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Handout 4 Statistical Physics for Communication and Computer Science
Homework 4 March 17, 2011, INR 113 - 9:15-11:00

Problem 1 (Stat phys formulation for the BEC channel)**.** We recall that the Binary Erasure Channel (BEC) is a memoryless channel with binary input alphabet $\{0, 1\}$ and output alphabet $\{0, 1, e\}$ where e is an erasure symbol. The channel transition probability is $p(0|0) = 1 - \epsilon$, $p(e|0) = \epsilon$, $p(1|1) = 1 - \epsilon$ and $p(e|1) = \epsilon$. Here you will specialize the formalism of last lecture to this case. Use the spin formulation for the bits $s_i = (-1)^{x_i}$ (i.e $s_i = +1$ when $x_i = 0$ and $s_i = -1$ when $x_i = 1$).

a) Show that the the channel distribution, expressed in terms of the likehood variables $h_i = \frac{1}{2} \ln \frac{p(y_i|+1)}{p(y_i|-1)}$, takes the form $c(h_i) = \epsilon\delta(h_i) + (1 - \epsilon)\Delta_\infty(h)$. Here $\Delta_\infty(h)$ represents a unit mass at infinity.

b) Consider a general Gibbs distribution with hamiltonian

$$\mathcal{H}(\vec{s}) = -\sum_C J_C \prod_{i \in C} s_i, \qquad C \subset \{1, ..., n\}, \qquad J_C \geq 0$$

Show that for such ferromagnetic systems (meaning $J_C \geq 0$) one always has

$$\langle s_i \rangle \geq 0$$

Hint: only the numerator of $\langle s_i \rangle$ might have a negative sign; show that in fact, this cannot the case by expanding the exponential. This is the simplest form of so-called Griffith inequalities for ferromagnetic systems.

c) Now we apply this general result to the BEC channel. Using the Nishimori identity $\mathbf{E}_{\vec{h}}[\langle s_i \rangle] = \mathbf{E}_{\vec{h}}[\langle s_i \rangle^2]$ proved in class (why is it true here ?), deduce that $\langle s_i \rangle$ is a random variable that takes values 0 or 1. Interpret this result in simple terms.

d) Condider now the EXIT function defined by $\frac{d}{d\epsilon} H(X^n|Y^n)$. Specialize the general formula given in class, namely

$$\frac{d}{d\epsilon} H(X^n|Y^n) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}_{\vec{h} \setminus h_i} \int dh_i \frac{dc(h_i)}{d\epsilon} \ln\left(\frac{1 + \langle s_i \rangle_{h_i=0} \tanh h_i}{1 + \tanh h_i} \right),$$

to the the BEC. In particular show that

$$\frac{d}{d\epsilon} H(X^n|Y^n) = \ln 2 \frac{1}{n} \sum_{i=1}^n (1 - \mathbb{E}_{\vec{h} \setminus h_i}[\langle s_i \rangle_{h_i=0}]) = \epsilon^{-1} \ln 2 \frac{1}{n} \sum_{i=1}^n (1 - \mathbb{E}_{\vec{h}}[\langle s_i \rangle]) \qquad (1)$$

Show that these quantities are directly related to the bit-MAP probability of error.

Problem 2. Consider the BIAWNC with noise variance σ^2 and the (binary) input normalized so that the $SNR = \sigma^{-2}$. In the last lecture we showed that

$$\frac{d}{d(\sigma^{-2})} \frac{1}{n} H(X^n|Y^n) = \frac{1}{2n} \sum_{i=1}^n (\mathbb{E}_{\vec{h}}[\langle s_i \rangle] - 1)$$

Use the same method to prove that

$$\frac{d^2}{d(\sigma^{-2})^2}\frac{1}{n}H(X^n|Y^n) = \frac{1}{n}\sum_{i,j=1}^{n}(\mathbb{E}_{\vec{h}}[(\langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle)^2])$$

Hint: you will have to use general Nishimori identities discussed in class.