Last time you proved that the Ising model in one dimension ($d = 1$) does not have a phase transition for any $T > 0$. On the grid $\mathbb{Z}^d$ there is a non trivial phase diagram with first and second order phase transitions for any $d \geq 2$. This is also the case on the complete graph (as shown in the lectures) which morally corresponds to $d = +\infty$. Another graph that in a sense, corresponds to $d = +\infty$, is the $q$-ary tree. Indeed on $\mathbb{Z}^d$ the number of lattice sites at distance less than $n$ from the origin scales as $n^d$. On the $q$-ary tree it scales as $(q - 1)^n$ which grows faster than $n^d$ for any finite $d$.

The goal of the two exercises below is to solve for the Ising model on a $q$-ary tree and show that it displays first and second order phase transitions (with similar qualitative properties than on a complete graph).

Consider a finite rooted tree and call the root vertex $o$. All vertices have degree $q$, except for the leaf nodes that have degree 1. We suppose that the tree has $n$ levels (the root being “level 0”). The thermodynamic limit corresponds to $n \to +\infty$. The Hamiltonian (multiplied by $\beta$) is

$$\beta \mathcal{H}_n = -K \sum_{(i,j) \in E_n} s_i s_j - h \sum_{i \in V_n} s_i$$

By $K > 0$, $h \in \mathbb{R}$, $V_n$ is the set of vertices and $E_n$ the set of edges. We are interested in the magnetization of the root node in the thermodynamic limit:

$$m(K,h) = \lim_{n \to +\infty} \langle s_o \rangle_n = \frac{\sum_{\{s_k \in V_n\}} s_o e^{-\beta \mathcal{H}_n}}{Z_n}$$

The formula $\tanh^{-1} y = \frac{1}{2} \ln \frac{1+y}{1-y}$ might be useful.

**Problem 1** (Recursive equations). Perform the sums over the spins attached at the leaf nodes and show that

$$\langle s_o \rangle_n = \frac{\sum_{\{s_k \in V_{n-1}\}} s_o e^{-\beta \mathcal{H}'_{n-1}}}{Z_{n-1}}$$

where $E_{n-1}$ and $V_{n-1}$ are the edge and vertex sets of a tree with with $n - 1$ levels and the new Hamiltonian is

$$\beta \mathcal{H}'_n = -K \sum_{(i,j) \in E_{n-1}} s_i s_j - h \sum_{i \in V_{n-1}} s_i - (q - 1) \tanh^{-1}(\tanh K \tanh h) \sum_{i \in \text{level } n-1} s_i$$

Iterate this calculation and deduce

$$\langle s_o \rangle_n = \tanh(h + q \tanh^{-1}(\tanh K \tanh u_n))$$

where

$$u_{k+1} = h + (q - 1) \tanh^{-1}(\tanh K \tanh u_k), \quad u_1 = h$$

Check that for $q = 2$ you get back the recursion of homework 2.
Problem 2 (Analysis of the recursion). We want to analyze the fixed point equation for $q \geq 3$,

$$u = h + (q - 1) \tanh^{-1}(\tanh K \tanh u) \quad (7)$$

Plot the curves $u \rightarrow u - h$ and $u \rightarrow (q - 1) \tanh^{-1}(\tanh K \tanh u)$ and show that:

- for $K \leq K_c \equiv \frac{1}{2} \ln \frac{q}{q-2} = \tanh^{-1}(q-1)^{-1}$, (7) has a unique solution, and that the iterations (6) converge to this unique solution.

- for $K > K_c$:
  - for $|h| \geq h_s$, (7) has a unique solution (you do not need to compute $h_s$ explicitly although it is possible to find its analytical expression) and that the iterations (6) converge to this unique solution.
  - for $|h| < h_s$, (7) has three solutions $u_-(h) < u_0(h) < u_+(h)$. Check graphically that for $h > 0$ the iterations (6) with initial condition $u_1 = h$ converge to $u_+(h)$. Similarly for $h < 0$ they converge to $u_-(h)$. Check also graphically that the fixed point $u_0(h)$ is unstable whereas $u_\pm(h)$ are stable.

Problem 3 (Phase transitions). Now we want to discuss the consequences of the results in problem 2 for the phase diagram. In a nutshell: in the $(K^{-1}, h)$ plane there is a first order phase transition line $(K^{-1} \in [0, K_c^{-1}[ , h = 0)$ terminated by a critical point $K_c$. Outside of this line $m(K, h)$ is an analytic function of each variable.

We define the ”spontaneous magnetization” as $m_{\pm}(K) = \lim_{h \to 0, \pm} m(K, h)$.

- Deduce from the analysis in problem 2 that for $K \leq K_c$, $m_+(K) = m_-(K) = 0$.

- Deduce that for $K > K_c$, $m_+(K) \neq m_-(K)$ (jump discontinuity or first order phase transition) and that for $K \to +\infty$ $m_{\pm} \to \pm 1$.

- Show that for $K \to K_c$ from above, $m_{\pm}(K) \sim (K - K_c)^{1/2}$. So on the line $h = 0$, as a function of $K$, the spontaneous magnetization is continuous but not differentiable at $K_c$ (second order phase transition).

- Now fix $K = K_c$ and show that $m(K_c, h) \sim |h|^{1/3}$. As a function of $h$ the spontaneous magnetization is continuous but not differentiable at $K_c$ (second order phase transition).

Hint: for the last two questions you can expand the fixed point equation to order $u^3$.

Remark: Note that the exponents 1/2 and 1/3 are the same than for the model on a complete graph. This is also the case for all $d \geq 4$ and is not the case for $d = 2, 3$. 
