ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 10 Statistical Physics for Communication and Computer Science

Homework 10 May 5, 2011, INR 113 - 9:15-11:00

Upper bounds for the SAT-UNSAT threshold, we call it α_s, are usually derived by counting arguments. The first exercise develops the simplest such argument. In the second exercise you will study a more subtle counting argument which leads to an important improvement[1]. This method can be further refined and has led to better bounds.

An assignment is a tuple $\underline{x} = (x_1, \ldots, x_n)$ where $x_i = 0, 1$ of n variables. The total number of possible clauses with k variables is equal to $2^k \binom{n}{k}$. A random formula F is constructed by picking, with replacement, uniformly at random, m clauses. Thus there are $(2^k \binom{n}{k})^m$ possible formulas.

We set $m = \alpha n$ and think of n and m as tending to ∞ with α fixed. This is the regime displaying a SAT-UNSAT threshold.

It is useful to keep in mind that $\mathbb{P}[A] = \mathbb{E}[1(A)]$ where $1(A)$ is the indicator function of event A. In what follows probabilities and expectations are with respect to the random formulas F.

Problem 1 (Crude upper bound by counting all satisfying assignments). Let $S(F)$ be the set of all assignments satisfying F and let $|S(F)|$ be its cardinality. Since F is a random formula, $|S(F)|$ is an integer valued random variable.

a) Show the Markov inequality $\mathbb{P}[F \text{ satisfiable}] \leq \mathbb{E}[|S(F)|]$.

b) Fix an assignement $\underline{x}$. Show that $\mathbb{P}[\underline{x}\,\text{satisfies}\,F] = (1 - 2^{-k})^m$. Then deduce that

$$\mathbb{E}[|S(F)|] = 2^n (1 - 2^{-k})^m.$$

c) Deduce the upper bound

$$\alpha_s < \frac{\ln 2}{|\ln(1 - 2^{-k})|}.$$

For $k = 3$ this yields $\alpha_s < 5.191$.

Problem 2 (Bound by counting a restricted set of assignments). We define the set $S_{\rm m}(F)$ of *maximal* satisfying assignments as follows. An assignment $\underline{x} \in S_{\rm m}(F)$ iff:

- $\underline{x}$ satisfies F,
- for all i such that $x_i = 0$ (in $\underline{x}$), the *single flip* $x_i \to 1$ yields an assignment - call it $\underline{x}^i$ - that *violates* F.

a) Show that if F is satisfiable then $S_m(F)$ is not empty. *Hint*: proceed by contradiction.

b) Show as in the first exercise the Markov inequality $\mathbb{P}[F \text{ satisfiable}] \leq \mathbb{E}[|S_m(F)|]$

c) Show that

$$\mathbb{E}[|S_m(F)|] = (1 - 2^{-k})^m \sum_{\underline{x}} \mathbb{P}[\cap_{i:x_i=0} (\underline{x}^i \text{ violates}\, F) \mid \underline{x}\,\text{satisfies}\, F].$$

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[1]by Kirousis, Kranakis, Krizanc and Stamatiou, *Approximating the Unsatisfiability Threshold of Random Formulas*, in Random Struct and Algorithms (1998).

d) Fix $\underline{x}$. The events $E_i \equiv (\underline{x}^i \,\text{violates}\, F)$ are negatively correlated, i.e

$$\mathbb{P}[\cap_{i:x_i=0} E_i \mid \underline{x}\,\text{satisfies}\, F] \leq \prod_{i:x_i=0} \mathbb{P}[E_i \mid \underline{x}\,\text{satisfies}\, F]$$

For the full proof which uses a correlation inequality (of FKG type) we refer to the reference given above. Here is a rough intuition for the inequality. First note that if $x_i = 0$ and $\underline{x}^i$ violates F, there must be some set S_i of clauses (in F) that are satisfied *only* by this variable $x_i = 0$ (this set might contain only one clause). This restricts the possible formulas contributing to the event E_i. Second note that sets S_i, S_j corresponding to different such variables $x_i = 0$, $x_j = 0$ must be *disjoint*. This "repulsion" between the sets S_i and S_j puts even more restrictions on the possible formulas, compared to a hypothetical situation where the events (and thus the sets S_i and S_j) would have been independent.

e) Now show that

$$\mathbb{P}[E_i \mid \underline{x}\,\text{satisfies}\, F] = 1 - \left(1 - \frac{\binom{n-1}{k-1}}{(2^k - 1)\binom{n}{k}}\right)^m.$$

Hint: note that in the event E_i there must be at least one clause containing $x_i = 0$ and containing other variables that do not satisfy it.

f) Deduce from the above results that $\lim_{n\to 0} \mathbb{P}[F \text{ satisfiable}] = 0$ as long as α satisfies

$$(1 - 2^{-k})^{\alpha}(2 - e^{-\frac{\alpha k}{2^k - 1}}) < 1.$$

The improvement compared with the first exercise resides in the factor $e^{-\frac{\alpha k}{2^k-1}}$. A numerical evaluation for $k = 3$ yields the bound $\alpha_s < 4.667$.