Problem 1 (The world of Ideals). We want to derive a relation between $X(e^{j2\pi f})$ and the DTFT of the downsampled signal, i.e $X_d(e^{j2\pi f})$. We do this in two steps.

Step 1: Consider first the signal

$$x_p[n] = \begin{cases} x[n] & n = 4k, \ k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Now, observe that

$$x_p[n] = \frac{1}{4} \left( x[n] + (e^{j\frac{2\pi}{4}})^n x[n] + (e^{j\frac{2\pi}{4}2})^n x[n] + (e^{j\frac{2\pi}{4}3})^n x[n] \right)$$

$$= \frac{1}{4} \sum_{k=0}^{3} e^{j\frac{2\pi}{4}kn} x[n]$$

Note that the complex exponentials are periodic with period 4, so it is sufficient to check that the above relation holds for $n = 0, 1, 2, 3$.

Using the frequency shift property of the DTFT, we get

$$X_p(e^{j2\pi f}) = \frac{1}{4} \sum_{k=0}^{3} X(e^{j(2\pi f - \frac{2\pi}{4}k)})$$

Step 2: Removing all zeros introduced in $x_p[n]$, we get $x_d[n]$. Hence,

$$x_d[n] = x_p[4n]$$

So,

$$X_d(e^{j2\pi f}) = \sum_{n \in \mathbb{Z}} x_d[n] e^{-j2\pi fn}$$

$$= \sum_{n \in \mathbb{Z}} x_p[4n] e^{-j2\pi fn}$$

$$= \sum_{m=4n, n \in \mathbb{Z}} x_p[m] e^{-j2\pi f \frac{m}{4}}$$

$$= (a) \sum_{m \in \mathbb{Z}} x_p[m] e^{-j2\pi f \frac{m}{4}}$$

$$= X_p(e^{j\frac{2\pi f}{4}})$$

Where (a) follows from the fact that $x_p[m] = 0$, for $m$ integer multiple of 4.

So overall, we obtained

$$X_d(e^{j2\pi f}) = \frac{1}{4} \sum_{t=0}^{3} X(e^{j\frac{2\pi f}{4} - \frac{2\pi}{4}t})$$
Now, we want to derive a relationship between $X_d(e^{j2\pi f})$ and the DTFT of the upsampled signal, i.e $X_u(e^{j2\pi f})$.

$$X_u(e^{j2\pi f}) = \sum_{n \in \mathbb{Z}} x_u[n]e^{-j2\pi fn}$$

$$= \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} x_d[k] \delta[n - 4k] \right) e^{-j2\pi fn}$$

$$= \sum_{k \in \mathbb{Z}} x_d[k] \sum_{n \in \mathbb{Z}} \delta[n - 4k] e^{-j2\pi fn}$$

$$= \sum_{k \in \mathbb{Z}} x_d[k] e^{-j2\pi f4k}$$

$$= X_d(e^{j2\pi f4})$$

The cascade of downsampling and upsampling operations yields:

$$X_u(e^{j2\pi f}) = \frac{1}{4} \sum_{k=0}^{3} X(e^{j(2\pi f - \frac{2\pi}{4}k)})$$

The signals are drawn in figures 1-5 at the end of the solutions.

**Problem 2** (Fractional Delay).

a) The system represents a "fractional" delay. Hence,

$$y[n] = \sin(2\pi f_0(n - d) + \phi_0)$$

i.e. $y[n]$ is "delayed" by $d$ time units. The simplest way to compute $y[n]$ is to transform into the Fourier domain, to multiply, and to transform back.

b) We have $h[n] = \delta[n - d]$. Since this impulse response is absolutely summable, the system is BIBO stable. If $d \geq 0$ then the system is causal, otherwise it is not.

c) If $d$ is not an integer, then we get by direct integration :

$$h[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi f d} e^{j2\pi fn}$$

$$= \frac{1}{2j\pi(n - d)} e^{j2\pi(n - d)}|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \text{sinc}(n - d)$$

In this case the impulse response is not absolutely summable, so the system is not BIBO stable. The system is never causal.
Problem 3.

\[ H(z) = \frac{(1 - \frac{1}{3}z^{-1})}{(1 + \frac{1}{2}z^{-1})(1 - z^{-1} + \frac{1}{2}z^{-2})} \]

1) \[ H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{3}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2} + \frac{1}{4}z^{-3}} = \frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-3}} \]

The difference equation is:

\[ y[n] - \frac{1}{2}y[n - 1] + \frac{1}{4}y[n - 3] = x[n] - \frac{1}{3}x[n - 1] \]

2) Note that the poles are located on the circles \(|z| = \frac{1}{2}\) and \(|z| = \frac{1}{\sqrt{2}}\). See Fig. 6.

We can associate three regions with \(H(z)\).

\(\text{ROC}_1: |z| > \frac{1}{\sqrt{2}}\). See Fig. 7.
\(\text{ROC}_2: |z| < \frac{1}{2}\). See Fig 8.
\(\text{ROC}_3: \frac{1}{2} < |z| < \frac{1}{\sqrt{2}}\). See Fig 9.

\(\text{ROC}_1\) includes the unit circle, so it is a stable system. Moreover, \(\text{ROC}_1\) extends outward from \(|z| = \frac{1}{\sqrt{2}}\) including \(\infty\). Hence it is a causal system.

On the other hand \(\text{ROC}_2\) does not include the unit circle, hence it is not stable. Since \(\text{ROC}_2\) is the inside of a circle, it cannot be causal either. Similarly, \(\text{ROC}_3\) is neither stable, nor causal.

3) The Fourier transform converges in \(\text{ROC}_1\), see Fig. 10.

4) \[ H(z) = \frac{(1 - \frac{1}{3}z^{-1})}{(1 + \frac{1}{2}z^{-1})(1 - z^{-1} + \frac{1}{2}z^{-2})} \]

\(\text{ROC}_1: |z| > \frac{1}{\sqrt{2}}\)

\[ \frac{(1 - \frac{1}{3}z^{-1})}{(1 + \frac{1}{2}z^{-1})(1 - z^{-1} + \frac{1}{2}z^{-2})} = \frac{A}{(1 + \frac{1}{2}z^{-1})} + \frac{B + Cz^{-1}}{(1 - z^{-1} + \frac{1}{2}z^{-2})} \]

\[ H(z)(1 + \frac{1}{2}z^{-1})|_{z = -\frac{1}{2}} = \frac{1 - \frac{1}{3}(-2)}{1 - (-2) + \frac{1}{2}(-2)^2} = \frac{1}{3} \]

\[ 1 - \frac{1}{3}z^{-1} = A \left( 1 - z^{-1} + \frac{1}{2}z^{-2} \right) + \left( 1 + \frac{1}{2}z^{-1} \right) (B + Cz^{-1}) \]

\[ = (A + B) + \left( -A + C + \frac{B}{2} \right) z^{-1} + \left( \frac{A}{2} + \frac{C}{2} z^{-2} \right) \]
\[ \Rightarrow B = \frac{2}{3} \]
\[ \Rightarrow C = -\frac{1}{3} \]

Hence, the partial fraction expansion gives:

\[
H(z) = \frac{\frac{1}{3}}{1 + \frac{1}{2}z^{-1}} + \frac{\frac{2}{3}(1 - \frac{1}{2}z^{-1})}{1 - z^{-1} + \frac{1}{2}z^{-2}}
\]

Since \( H(z) \) is causal inside \( \text{ROC}_1 \), the inverse is given by:

\[
h[n] = \frac{1}{3} \left( -\frac{1}{2} \right)^n u[n] + \frac{2}{3} \left( \frac{1}{\sqrt{2}} \right)^n \cos \left( \frac{\pi}{4} n \right) u[n]
\]

\( \text{ROC}_2: |z| < \frac{1}{2} \)

You need further to expand the second fraction with partial fraction expansion.

\[
\frac{(1 - \frac{1}{2}z^{-1})}{(1 - z^{-1} + \frac{1}{2}z^{-2})} = \frac{D}{(1 - \sqrt{2}/2 e^{(j\pi/4)}z^{-1})} + \frac{E}{(1 - \sqrt{2}/2 e^{(-j\pi/4)}z^{-1})}
\]

\[ \Rightarrow D = \frac{1}{2} \]
\[ \Rightarrow E = \frac{1}{2} \]

So

\[
H(z) = \frac{\frac{1}{3}}{1 + \frac{1}{2}z^{-1}} + \frac{\frac{2}{3}D}{(1 - \sqrt{2}/2 e^{(j\pi/4)}z^{-1})} + \frac{\frac{2}{3}E}{(1 - \sqrt{2}/2 e^{(-j\pi/4)}z^{-1})}
\]

Since \( H(z) \) is anti-causal inside \( \text{ROC}_2 \), the inverse is given by:

\[
h[n] = -\frac{1}{3} \left( -\frac{1}{2} \right)^n u[-n - 1] - \frac{2}{3}D \left( \frac{\sqrt{2}}{2} e^{(j\pi/4)} \right)^n u[-n - 1] - \frac{2}{3}E \left( \frac{\sqrt{2}}{2} e^{(-j\pi/4)} \right)^n u[-n - 1]
\]

\[
= -\frac{1}{3} \left( -\frac{1}{2} \right)^n u[-n - 1] - \frac{2}{3} \text{Re} \left\{ \left( \frac{\sqrt{2}}{2} e^{(j\pi/4)} \right)^n \right\} u[-n - 1]
\]

\( \text{ROC}_3: \frac{1}{2} < |z| < \frac{1}{\sqrt{2}} \)

Using the previous partial fraction expansion, we see that the first fraction of \( H(z) \) is causal, the other two fractions are anti-causal. Hence,

\[
h[n] = \frac{1}{3} \left( -\frac{1}{2} \right)^n u[n] - \frac{2}{3} \text{Re} \left\{ \left( \frac{\sqrt{2}}{2} e^{(j\pi/4)} \right)^n \right\} u[-n - 1]
\]

5) We want to find \( G(z) \) such that \( H(z)G(z) = 1 \)

i)

\[
G(z) = \frac{\left( 1 + \frac{1}{2}z^{-1} \right) \left( 1 - z^{-1} + \frac{1}{2}z^{-2} \right)}{\left( 1 - \frac{1}{2}z^{-1} \right)}
\]

See Fig. 11. We found that \( H(z) \) is both stable and causal inside \( \text{ROC}_1 \). Similarly we can deduce that \( G(z) \) is both causal and stable for the region \( \text{ROC}_G: |z| > \frac{1}{3} \). Now, we also need to ensure that \( \text{ROC}_1 \cap \text{ROC}_G \neq \emptyset \), which holds in this case. The DTFT is plotted in Fig. 12.
Figure 6: Pole-zero plot

Figure 7: ROC₁ : \( |z| > \frac{1}{\sqrt{2}} \)

Figure 8: ROC₂ : \( |z| < \frac{1}{2} \)
Figure 9: ROC_3: \( \frac{1}{2} < |z| < \frac{1}{\sqrt{2}} \)

Figure 10: Magnitude response of \(|H(e^{j2\pi f})|\)
Figure 11: Pole-zero plot with ROC for $G(z)$

Figure 12: Magnitude response for $G(z)$
Problem 4. (FIR Approximation of the Hilbert Filter/Oppenheim Problems 7.32/7.33/7.52)

1) i) 

\[ H_s(e^{j2\pi f}) = \sum_{n=0}^{M} h_s[n]e^{-j2\pi fn} \]

\[ = \sum_{n=0}^{M-1} h_s[n]e^{-j2\pi fn} + \sum_{n=M+1}^{M} h_s[n]e^{-j2\pi fn} \]

\[ = \sum_{n=0}^{M-1} h_s[n]e^{-j2\pi fn} + \sum_{m=0}^{M-1} h_s[M-M]e^{-j2\pi f(M-m)} \]

\[ = e^{-j2\pi f M} \sum_{n=0}^{M-1} h_s[n]e^{j2\pi f(M-n)} + \sum_{n=0}^{M-1} h_s[n]e^{-j2\pi f(M-n)} \]

\[ = e^{-j2\pi f M} \sum_{n=0}^{M-1} 2h_s[n] \cos \left(2\pi f \left(\frac{M}{2} - n\right)\right) \]

\[ = e^{-j2\pi f M} \sum_{n=1}^{M+1} 2h_s \left[\frac{M+1}{2} - n\right] \cos \left(2\pi f \left(n - \frac{1}{2}\right)\right) \]

ii) Similarly, by considering the fact that \( h[n] = 0 \) for \( n < 0 \) and \( n > M \), and the fact that \( h[n] = h[M-n] \) for \( n = 0, ..., \frac{M-1}{2} \), we can derive:

\[ H_{as}(e^{j2\pi f}) = j e^{j2\pi f \frac{M}{2}} \sum_{k=1}^{M+1} 2h_{as} \left[\frac{M+1}{2} - k\right] \sin \left(2\pi f \left(k - \frac{1}{2}\right)\right) \]

2) i) The Hilbert transform is given by:

\[ H_h(e^{j2\pi f}) = \begin{cases} e^{j\frac{\pi}{2}} & -\frac{1}{2} < f < 0 \\ e^{-j\frac{\pi}{2}} & 0 < f < \frac{1}{2} \end{cases} \]

Hence, the delayed Hilbert transform with generalized linear phase can be defined as:

\[ H_d(e^{j2\pi f}) = \begin{cases} e^{j\frac{\pi}{2} - j2\pi fd} & -\frac{1}{2} < f < 0 \\ e^{-j\frac{\pi}{2} - j2\pi fd} & 0 < f < \frac{1}{2} \end{cases} \]

The magnitude and phase responses are plotted in Fig. 13, and Fig. 14 respectively.

ii) Note that the phase response has a \( \pi \) radian phase shift at \( f = 0 \). This is because the above Hilbert filter requires a zero at \( z = 1 \). This implies that the filter coefficients sum up to 0. Hence the filter should be antisymmetric. So we could only use \( h_{as}[n] \) to approximate \( h_d[n] \).
iii) 

\[ h_d[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} H_d(e^{j2\pi f})e^{j2\pi fn} df \]
\[ = \int_{-\frac{1}{2}}^{0} e^{j(\frac{n}{2}-2\pi fd)}e^{j2\pi fn} df + \int_{0}^{\frac{1}{2}} e^{-j(\frac{n}{2}+2\pi fd)}e^{j2\pi fn} df \]
\[ = e^{j\frac{n}{2}} \int_{-\frac{1}{2}}^{0} e^{j2\pi f(n-d)} df + e^{-j\frac{n}{2}} \int_{0}^{\frac{1}{2}} e^{j2\pi f(n-d)} df \]
\[ = \begin{cases} \frac{1}{\pi(n-d)}[1 + \sin(\pi(n-d) - \frac{\pi}{2})] & n \neq d \\ 0 & n = d \end{cases} \]
\[ = \begin{cases} \frac{1}{\pi(n-d)}[1 - \cos(\pi(n-d))] & n \neq d \\ 0 & n = d \end{cases} \]
\[ = \begin{cases} \frac{2\sin^2(\frac{\pi}{2}(n-d))}{\pi(n-d)} & n \neq d \\ \pi(n-d) & n = d \end{cases} \]

We observe that \( h_d[n] \) is symmetric around \( n = d \). Moreover, from part (ii) we also know that \( h_{as}[n] \) can be used to approximate \( h_d[n] \) as a causal, FIR filter with generalized linear phase. Hence \( d = \frac{M}{2} \) since \( \frac{M}{2} \) is the axis of symmetry of \( h_{as}[n] \).

iv) From Parseval, we have

\[ e^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |H(e^{j2\pi f}) - H_d(e^{j2\pi f})|^2 df = \sum_{n \in \mathbb{Z}} |h_d[n] - h[n]|^2 \]

We know that due to the windowing operation \( h[n] = 0 \) for \( n < 0 \) and \( n > M \). As a result to minimize \( e^2 \), the best thing we could do is to select \( h[n] = h_d[n] \) for \( 0 \leq n \leq M \). Therefore the optimal window which minimizes \( e^2 \) is the rectangular window, i.e :

\[ w[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases} \]
Problem 5. Note that the solution is given in terms of the "w" variable notation for DTFT!

\[ z[n] = x[n] * y[n], \quad z[n] = \frac{1}{2\pi} \int_{0}^{2\pi} 2e^{j\omega}e^{j\omega n} d\omega \]
\[ \Rightarrow Z(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega}). \]
\[ X(e^{j\omega}) = \frac{1}{2\pi} \sum_{n=-49}^{49} \frac{49-n}{49} e^{-j\omega n} \]
\[ Y(e^{j\omega}) = \sum_{n=49}^{49} 49 \sum_{n'=49}^{49} (49-n') e^{-j\pi\frac{n'}{49}} \]
\[ z[n] = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \sum_{n=-49}^{49} \frac{49-n}{49} e^{-j\omega n} \right) \frac{1}{2} 2\pi \left[ e^{j\pi\frac{n}{49}} (\omega - \frac{\pi}{49}) + e^{-j\pi\frac{n}{49}} (\omega + \frac{\pi}{49}) \right] e^{j\omega n} d\omega \]
\[ = \frac{1}{2\pi} \sum_{n=-49}^{49} \left( \sum_{n'=49}^{49} \frac{49-n'}{49} e^{-j\pi\frac{n'}{49}} \right) e^{j\pi\frac{n}{49}} 2\pi + \frac{1}{2\pi} \sum_{n=-49}^{49} \left( \sum_{n'=49}^{49} (49-n') e^{j\pi\frac{n'}{49}} \right) e^{-j\pi\frac{n}{49}} 2\pi \]

Let \( a = \sum_{n'=-49}^{49} \frac{49-n'}{49} e^{-j\pi\frac{n'}{49}} \) and \( a^* = \sum_{n'=-49}^{49} \frac{49-n'}{49} e^{j\pi\frac{n'}{49}} \). Then we have:

\[ z[n] = \frac{1}{2} e^{j\pi\frac{n}{49}} a e^{j\pi\frac{n}{49}} + \frac{1}{2} e^{j\pi\frac{n}{49}} a^* e^{-j\pi\frac{n}{49}} \]

Now we only need to compute \( a \).

\[ a = \sum_{n'=0}^{98} \frac{n'}{49} e^{-j\pi\frac{n'+49}{49}} = e^{-j\pi\frac{99}{49}} \sum_{n'=0}^{98} \frac{n'}{49} e^{j\pi\frac{n'}{49}} \]

We know from hw1 that

\[ \sum_{n=0}^{k} \alpha^n = \frac{1-\alpha^{k+1}}{1-\alpha} \]
\[ \sum_{k=1}^{n} k x^k = x \sum_{k=1}^{n} k x^{k-1} = x \sum_{k=1}^{n} \frac{dx^k}{dx} = x \frac{d}{dx} \left( \sum_{k=1}^{n} x^k \right) \text{ by linearity of differentiation} \]
\[ = x \frac{d}{dx} \left( \frac{1-x^{n+1}}{1-x} - 1 \right) = x \frac{-(n+1)x^n(1-x) - (1-x^{n+1})(-1)}{(1-x)^2} = x - (n+1)x^{n+1} + nx^{n+2} \]

So,

\[ a = \frac{e^{-j\pi\frac{99}{49}} e^{j\pi\frac{98}{49}} - (99)e^{j\pi\frac{99}{49}} + 98e^{j\pi\frac{100}{49}}}{(1-e^{j\pi\frac{99}{49}})^2} \]
Figure 1: Original DTFT

Figure 2: DTFT After Up and Down Sampling

Figure 3: DTFT Using the Low Pass Filter

Figure 4: DTFT Using the Band Pass Filter

Figure 5: DTFT Using the High Pass Filter

Figure 6: Magnitude Response of the Filter

Figure 7: Phase Response of the Filter