Problem 1 (Golay Sequences).

i) a) We have to show that $\sum_{i=0}^{N-k-1} a_i a_{i+k} + b_i b_{i+k} = 2N\delta[k]$.

$$\begin{aligned}
\sum_{i=0}^{N-k-1} a_i a_{i+k} + b_i b_{i+k} &= \sum_{i=0}^{N-k-1} (-1)^i x_i (-1)^{i+k} x_{i+k} + (-1)^i y_i (-1)^{i+k} y_{i+k} \\
&= \sum_{i=0}^{N-k-1} (-1)^k (x_i x_{i+k} + y_i y_{i+k}) \\
&= (-1)^k \sum_{i=0}^{N-k-1} (x_i x_{i+k} + y_i y_{i+k}) \\
&= (-1)^k 2N\delta[k] \\
&= \begin{cases} 2N & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}
\end{aligned}$$

which implies that $R_a(k) + R_b(k) = 2N\delta[k]$.

b) We compute:

$$\begin{aligned}
\sum_{i=0}^{N-k-1} \hat{x}_i \hat{x}_{i+k} + y_i y_{i+k} &= \sum_{i=0}^{N-k-1} x_{N-i-1} x_{N-i-1-k} + \sum_{i=0}^{N-k-1} y_i y_{i+k} \\
&= \sum_{j=N-1-k}^{0} x_{j+k} x_j + \sum_{i=0}^{N-k-1} y_i y_{i+k} \\
&\overset{(a)}{=} \sum_{j=0}^{N-k-1} x_{j+k} x_j + y_j y_{j+k} \\
&= 2N\delta[k]
\end{aligned}$$

where (a) is due to a change of variable $j = N - i - 1 - k$.

ii) We have $X\left(e^{j2\pi f}\right) = \sum_{n=0}^{N-1} x_n e^{-j2\pi f n}$. Using Parseval theorem :

$$\sum_{n=-\infty}^{+\infty} x_n x_n^* = \sum_{n=0}^{N-1} x_n x_n^* = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|X(e^{j2\pi f})\right|^2 df$$

Moreover, we know that:

$$\sum_{n=0}^{N-1} x_n x_n^* = \sum_{n=0}^{N-1} x_n^2 = N$$

Therefore,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|X(e^{j2\pi f})\right|^2 df = N$$

This means that the average of $\left|X(e^{j2\pi f})\right|^2$ over the interval $[-\frac{1}{2}, \frac{1}{2}]$ is equal to N. So, for at least one value of $f \in [-\frac{1}{2}, \frac{1}{2}]$, it should be greater or equal to N. This proves that $max_f \left|X(e^{j2\pi f})\right|^2 \geq \sqrt{N}$.

iii) Since X and Y are complementary we have $\sum_{i=0}^{N-k-1} x_i x_{i+k} + y_i y_{i+k} = 2N\delta[k]$. Furthermore, since $x[n]$ is equal to zero for $n \geq N$, we have :

$$\sum_{i=0}^{N-k-1} x[i]x[i+k] + y[i]y[i+k] = \sum_{i=0}^{N-1} x[i]x[i+k] + y[i]y[i+k] = 2N\delta[k] \tag{1}$$

Now, consider the signal $\bar{x}[i] = x[-i]$ and $\bar{y}[i] = y[-i]$. Then,

$$\bar{X}\left(e^{j2\pi f}\right) = \sum_{-\infty}^{\infty} \bar{x}[n]e^{-j2\pi fn} = \sum_{-\infty}^{\infty} x[-n]e^{-j2\pi fn} = X\left(e^{-j2\pi f}\right) = X\left(e^{j2\pi f}\right)^*$$

Similarly, we have $\bar{Y}\left(e^{j2\pi f}\right) = Y\left(e^{j2\pi f}\right)^*$. Now,

$$R_x[k] = \sum_{i=0}^{N-1} x[i]x[i+k] = \sum_{i=0}^{N-1} x[i]\bar{x}[-i-k] = x[i] * \bar{x}[i]$$

which means that the autocorrelation function equal the convolution of $x[i]$ and $\bar{x}[i]$. Therefore,

$$R_x(e^{j2\pi f}) = X(e^{j2\pi f})\bar{X}(e^{j2\pi f}) = \left|X(e^{j2\pi f})\right|^2$$

Similarly, we have $R_y(e^{j2\pi f}) = \left|Y(e^{j2\pi f})\right|^2$. Finally from (1), we get

$$R_x(e^{j2\pi f}) + R_y(e^{j2\pi f}) = 2N$$
$$\left|X(e^{j2\pi f})\right|^2 + \left|Y(e^{j2\pi f})\right|^2 = 2N$$

which implies that $\left|X(e^{j2\pi f})\right|, \left|Y(e^{j2\pi f})\right| \leq \sqrt{2N}$.

Problem 2 (LTI Systems)**.**

i) $T(x[n]) = \sum_{k=n_0}^{n} x[k]u[n-k]$.
The system is *not stable* because the output is the summation of input samples from n_0 up to n and it can go to infinity even if the input is bounded. It is *linear* because summation is a linear operator. It is *causal* because $y[n]$ just depends on $x[n_0], x[n_0+1], \ldots, x[n]$ and not future samples such as $x[n+1], \ldots$. It is *not memoryless* because as you can see $y[n]$ depends on some other input samples which are not for the present time. It is *not time invariant* , because if for example $x_1[n] = u[-n+n_0-1]$ where $u[n]$ is a step function, the output will be $y_1[n] = 0$ for $n > n_0$, while if we shift the input by 1 to the right, the output will be $y_2[n] = 1$ for $n > n_0$, which is not the shifted version of $y_1][n]$.

ii) $T(x[n]) = x[Mn]$.
This system is *stable* and *linear* trivially, because $|x[Mn]| < \infty \rightarrow |y[n]| < \infty$ and $T[ax_1[n] + bx_2[n]] = ax_1[Mn] + bx_2[Mn] = aT[x_1[n]] + bT[x_2[n]]$. It is *not causal* because $y[n]$ depends on future signal samples (M is positive). So, it is also *not memoryless* . Let's check if it is time invariant or not. If $x_2[n] = x_1[n - n_d]$, then, $y_2[n] = x_2[Mn] = x_2[Mn - n_d]$ while $y_1[n - n_d] = x_1[M(n - n_d)] = x_1[Mn - Mn_d]$ which is not equal to $y_1[n - n_d]$. So, it is *not time invariant.*

iii) $T(x[n]) = x[n] * x[n]$.
The system is *not linear* because $T(2x[n]) = 4T(x[n]) \neq 2T(x[n])$. It is *not time invariant* because if we define $x_1[n] = x[n-n_d]$, then $y_1[n] = T(x_1[n]) = \sum_{k=-\infty}^{\infty} x[k]x[n-k-2n_d] \neq y[n-n_d] = \sum_{k=-\infty}^{+\infty} x[k]x[n-k-n_d]$. Since the output at time n depends both on previous and future values of n the system is *neither memoryless, nor causal.* Finally as in part (i), the system is *not stable.*

iv) $T(x[n]) = median\{x[n - M_1], \ldots, x[n], \ldots, x[n + M_2]\}$.
We show the system is *not linear* with a counter-example. Take $a = 1$, $b = 1$, and $x_1[n] = \{0, 1, 5\}$, $x_2[n] = \{6, -1, -2\}$. Then $T(ax_1[n] + bx_2[n]) = median\{6, 0, 3\} = 3 \neq aT(x_1[n]) + bT(x_2[n]) = median\{0, 1, 5\} + median\{6, -1, -2\} = 1 + -1 = 0$. The system is *time invariant* because if we let $x_1[n] = x[n - n_d]$. Then $y_1[n] = med\{x1[n-M_1] \ldots x1[n+M_2]\} = med\{x[n-n_d-M_1] \ldots x[n-n_d+M_2]\} = y[n-n_d]$. Since the output depends on previous and future values of the input, the system is *neither memoryless nor causal* in general. However, if $M_1 = M_2 = 0$, then the system is memoryless as $T(x[n]) = x[n]$ and if only $M_2 = 0$, then the system is causal. The system is stable because each value of the output equals some input value. Hence, if the input is bounded for all n, the output will be too.

v) $T(x[n]) = x[n] * h_1[n] * h_2[n] * h_3[n]$.
This is a cascaded system. First note that if n is not a multiple of M, $T(x[n] = 0)$. If n is a multiple of M, we can write the overall transformation as $T(x[n]) = x[n] - \frac{1}{2}x[n - M]$. It is easily verified that the system is *linear* and *time invariant* . For $M \neq 0$ it is *not memoryless* . Since the output only depends on the current and previous inputs, the system is *causal* . It is *stable* since the output will be bounded for a bounded input.

Problem 3 (Z-Transform)**.**

i)

$$X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{+\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}$$

$$\text{for } |az^{-1}| < 1 \Rightarrow |z| > |a|$$

ii)

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-1} = \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{n=1}^{\infty} -a^{-n} z^n \\
&= -\sum_{n=1}^{\infty} (a^{-1}z)^n = -\left(\frac{1}{1-a^{-1}z} - 1\right) \text{ for } |a^{-1}z| < 1 \\
&= 1 - \frac{1}{1-a^{-1}z} = \frac{1-a^{-1}z-1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} \text{ for } |z| < |a|
\end{aligned}$$

We note that $x[n] = a^n u[n]$ is a causal sequence, and $x[n] = -a^n u[-n-1]$ is an anti-causal sequence.

Problem 4 (Inverse Z-Transform)**.**

1)

i) $X(z) = z + \dfrac{(1-z^{-2})(1+\frac{1}{2}z^{-1})}{(1+z)}$, $0 < |z| < \infty$. Note that $z = -1$ is not a pole of $X(z)$.

$$\begin{aligned}
X(z) &= z + \frac{(1-z^{-1})(1+z^{-1})(1-\frac{1}{2}z^{-1})}{z(1+z^{-1})} \\
&= z + z^{-1}(1-z^{-1})(1-\frac{1}{2}z^{-1}) \\
&= z + (z^{-1}-z^{-2})(1-\frac{1}{2}z^{-1}) \\
&= z + z^{-1} - \frac{1}{2}z^{-2} - z^{-2} + \frac{1}{2}z^{-3} \\
&= z + z^{-1} - \frac{3}{2}z^{-2} + \frac{1}{2}z^{-3} \text{ , for } 0 < |z| < \infty
\end{aligned}$$

Which gives :

$$x[n] = \begin{cases} 1 & n = -1 \\ 1 & n = 1 \\ -\frac{3}{2} & n = 2 \\ \frac{1}{2} & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

ii) $X(z) = \log(1-2z^{-1})$, $|z| > 2$..

Using the Taylor expansion we get :

$$\begin{aligned}
X(z) &= -\sum_{n=1}^{\infty} \frac{(2z^{-1})^n}{n}, |z| > 2 \\
&= -\sum_{n=1}^{\infty} \frac{2^n}{n} z^{-n}, |z| > 2
\end{aligned}$$

Hence,

$$x[n] = -\frac{2^n}{n} u[n-1]$$

2)

i) $X(z) = \dfrac{1}{(1-\frac{1}{7}z^{-1})(1-5z^{-1})}, \qquad \dfrac{1}{7} < |z| < 5$

We have two poles at $z_1 = \dfrac{1}{7}$ and $z_2 = -5$. So,

$$X(z) = \frac{\frac{1}{36}}{\left(1-\frac{1}{7}z^{-1}\right)} + \frac{\frac{35}{36}}{(1+5z^{-1})}, \frac{1}{7} < |z| < 5$$

The first part of $X(z)$ is causal, the second part is anti-causal. Hence,

$$x[n] = \frac{1}{36}(\frac{1}{7})^n u[n] - \frac{35}{36}(-5)^n u[-n-1].$$

ii) Note that the degree of the numerator is equal to the degree of the denominator. So we first use polynomial division and then apply partial fraction expansion.
The polynomial division gives,

$$X(z) = \frac{1+\frac{1}{2}z^{-2}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} = 4(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}) + (-3+3z^{-2}), |z| > \frac{1}{2}$$

And then we have

$$X(z) = 4 + \frac{-3(1-z^{-2})}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}$$

The second part have two poles at $z_1 = \dfrac{1}{4}$ and $z_2 = \dfrac{1}{2}$. Hence,

$$X(z) = 4 + \frac{-9}{(1-\frac{1}{4}z^{-1})} + \frac{6}{(1-\frac{1}{2}z^{-1})}, |z| > \frac{1}{2}$$

The system is causal, hence

$$x[n] \;=\; 4\delta[n] - 9\left(\frac{1}{4}\right)^n u[n] + 6\left(\frac{1}{2}\right)^n u[n].$$

3) Let us first rearrange the function,

$$X(z) = \frac{\left(1-\frac{1}{2}z^{-1}\right)}{(1-\frac{1}{4}z^{-1})(1+z^{-1})} = \frac{z(z-\frac{1}{z})}{(z-\frac{1}{4})(z+1)} = \frac{A(z)}{B(z)}, |z| > 1$$

Hence, we get

$$\begin{aligned} x[n] &= \frac{1}{i2\pi}\oint_c X(z)z^{n-1}dz \\ &= Res(X(z)z^{n-1})|_{z=\frac{1}{4}} + Res(X(z)z^{n-1})|_{z=-1} \\ &= \left(-\frac{1}{5}\left(\frac{1}{4}\right)^n + \frac{6}{5}(-1)^n\right)u[n] \end{aligned}$$

(Note that we found the residues using $Res(X(z)z^{n-1})|_{z=z_i} = \frac{A(z_i)}{B'(z_i)}z_i^{n-1}$ in this case.)