ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 4 Signal Processing for Communications

Solution 2 March 7, 2011, INF213 – 10:15am-12:00

Problem 1 (Any Basis of a Hilbert Space has Same Cardinality)**.**

Since B is a basis for H, we can write all x_i' for $i = 1, \dots, m$ as

$$x_i' = \sum_{j=1}^{n} \alpha_{ij} x_j.$$

Consider $\langle x_k', x_l' \rangle$ for $k \neq l, k, l = 1, \dots, m$.

$$\begin{aligned}
\langle x_k', x_l' \rangle &= \langle \sum_{j=1}^{n} \alpha_{kj} x_j, \sum_{i=1}^{n} \alpha_{li} x_i \rangle && \text{using the distributive and scaling properties} \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_{kj} \alpha_{li}^* \langle x_j, x_i \rangle && \text{since } \langle x_j, x_i \rangle = 0, \text{for } i \neq j \quad \text{and } \langle x_i, x_i \rangle = 1 \\
&= \sum_{j=1}^{n} \alpha_{kj} \alpha_{lj}^* = 0.
\end{aligned}$$

If we define $(\alpha_{k1}, \dots, \alpha_{kn})$ as the vector $\bar{\alpha}_k \in \mathbb{C}^n$, then the above condition is equivalent to:

$$(*) \qquad \langle \bar{\alpha}_k, \bar{\alpha}_l \rangle = 0 \qquad \forall k, l = 1, \dots, m \text{ and } k \neq l.$$

Since any set of orthogonal vectors in $\mathbb{C}^n$ has cardinality at most n, we can have at most n vectors $\bar{\alpha}_i, i = 1, \dots, n$ which fulfills $(*)$. Hence $m \leq n$.

We can do the same for expanding $\{x_i\}$ in terms of the basis B', which implies that $n \leq m$. Therefore, $m = n$.

Problem 2 (Gram-Schmidt)**.**

In Gram-Schmidt procedure, we make an orthonormal basis from a given set of vectors $\{u_1, \cdots, u_n\}$. At each step, we pick vector u_l from the set and make an orthonormal vector that is orthogonal to the subspace of the already chosen vectors $\{u_1, \cdots, u_{l-1}\}$ with the following procedure.

We find the projection of u_l in the subspace and then reduce the projection from u_l. The resulting vector is orthogonal to the subspace and consequently to all previous vectors. After normalization, it is a new member of our basis. At the first step, we start by normalizing u_1 as the first element of the basis.

In this problem,

$$\begin{aligned}
v_1 &= \frac{u_1}{||u_1||} = \frac{1}{2}(1, -1, 1, -1), \\
w_2 &= u_2 - \langle u_2, v_1 \rangle v_1 = (5, 1, 1, 1) - \frac{2}{2}(1, -1, 1, -1) = (4, 2, 0, 2),
\end{aligned}$$

where $\langle u_2, v_1\rangle v_1$ is the projection of u_2 on v_1.
Then $v_2 = \frac{w_2}{||w_2||} = \frac{1}{\sqrt{24}}(4, 2, 0, 2)$.
$w_3 = u_3 - \langle u_3, v_1\rangle v_1 - \langle u_3, v_2\rangle v_2 = (-3, -3, 1, -3) - (1, -1, 1, -1) + (4, 2, 0, 2) = (0, 0, 0, 0)$.
Since $w_3 = 0$, it means that u_3 is in the subspace of $\{v_1, v_2\}$ and it does not introduce a new dimension. Therefore, these three vectors are in a space spanned by $\{v_1, v_2\}$.

Problem 3 (Various Norms)**.**

We should verify the three properties of a norm :

(i) strict positivity : $v(x) \geq 0$ and $v(x) = 0 \Leftrightarrow x = 0$

(ii) homogeneity : $v(\alpha x) = |\alpha| v(x)$

(iii) triangle inequality : $v(x + y) \leq v(x) + v(y)$

Let us first check $v_1(x)$:

(i) $v_1(x) = \sum_{k=1}^{N} |x_k| \geq 0$ since $|x_i| \geq 0$ for all i and
$v_1(x) = \sum_{k=1}^{N} |x_k| = 0$ if for all i, $|x_i| = 0$ which is $(0, 0, \dots, 0)$.

(ii) We know that if $y, z \in \mathbb{C}$ then $|y \cdot z| = |y||z|$. Therefore,
$v_1(\alpha x) = \sum_{k=1}^{N} |\alpha x_k| = \sum_{k=1}^{N} |\alpha||x_k| = |\alpha| \sum_{k=1}^{N} |x_k| = |\alpha| v_1(x)$.

(iii) Let y, z be two complex numbers. Then
$|y + z|^2 = (y + z)(y + z)^* = yy^* + yz^* + zy^* + zz^* = |y|^2 + |z|^2 + yz^* + zy^*$.
zy^* is the complex conjugate of yz^*. Therefore, $yz^* + zy^* = 2\,\mathrm{Re}\,\{yz^*\} \leq 2|yz^*|$ where $\mathrm{Re}\,\{\cdot\}$ denotes the real part of a complex number. Hence, $|yz|^2 \leq |y|^2 + |z|^2 + 2|y||z| = (|y| + |z|)^2$. It means that $|y + z| \leq |y| + |z|$. Thus, $v_1(x + y) = \sum_{k=1}^{N} |x_k + y_k| \leq \sum_{k=1}^{N} (|x_k| + |y_k|) = \sum_{k=1}^{N} |x_k| + \sum_{k=1}^{N} |y_k| = v_1(x) + v_1(y)$.

Therefore, $v_1(x)$ is a norm on $\mathbb{C}^N$.

We do the same for $v_2(x)$:

(i) $v_2(x) = (\sum_{k=1}^{N} |x_k|^2)^{1/2} \geq 0$ since for every k, $|x_k|^2 \geq 0$ and $v_2(x) = 0$ iff for all k, $x_k = 0$.

(ii) $v_2(\alpha x) = (\sum_{k=1}^{N} |\alpha x_k|^2)^{1/2} = (\sum_{k=1}^{N} |\alpha|^2 |x_k|^2)^{1/2} =)(|\alpha|^2 \sum_{k=1}^{N} |x_k|^2)^{1/2} = |\alpha| v_2(x)$.

(iii) To verify the triangle inequality, we use Minkowsky lemma :

$$(\sum_{k=1}^{\infty} |x_k + y_k|^p)^{1/p} \leq (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} + (\sum_{k=1}^{\infty} |y_k|^p)^{1/p}, \qquad p \geq 1.$$

This lemma is much more than what we need to prove the triangle inequality for special case $p = 2$. It says that for not only $p = 2$ and finite dimensional spaces, but also for any arbitrary $p \geq 1$ and infinite dimensional spaces the triangle inequality holds.

Problem 4 (Convergent Sequences are Cauchy Sequences)**.**

On a metric space with metric $d(\cdot,\cdot)$, a sequence x_n is convergent to x, if for every ε, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \qquad \text{for all } n > N.$$

We should show that every convergent sequence is a Cauchy sequence, i.e. for every ε, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \qquad \text{for all } m, n > N.$$

Assume that x_n converges to x. From triangular property of metrics :

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x)$$

Then since x_n converges to x, for every $\varepsilon/2$, there exists N such that

$$\left.\begin{cases} d(x_n, x) < \varepsilon/2 & n > N \\ d(x_m, x) < \varepsilon/2 & m > N \end{cases}\right\} \Rightarrow d(x_n, x_m) < \varepsilon \qquad \text{for all } n, m > N$$

Therefore, it is a Cauchy sequence.

Problem 5 (Incompleteness of $\mathbb{Q}$)**.**

1. a_{n+1} is positive if a_n is positive. As we started from $a_1 = 2$, then a_n is always positive. On the other hand :

$$a_{n+1} = \frac{a_n^2 + 2}{2a_n} \geq \sqrt{2} \text{ since } a_n^2 + 2 \geq 2\sqrt{2}a_n \Leftrightarrow (a_n - \sqrt{2})^2 \geq 0$$

Thus , a_n is bounded from below by $\sqrt{2}$. Moreover it is decreasing, since

$$a_{n+1} \leq a_n \Leftrightarrow \frac{1}{a_n} \leq \frac{a_n}{2} \Leftrightarrow a_n \geq \sqrt{2}$$

Therefore, a_n is a decreasing sequence bounded between $\sqrt{2}$ and 2. We know from monotone convergence theorem, that the monotone and bounded sequence in $\mathbb{R}$ with metric of absolute value is convergent. To find the limit, assume that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$. Hence :

$$L = \frac{L}{2} + \frac{1}{L} \Rightarrow \frac{L}{2} = \frac{1}{L} \Rightarrow L = \sqrt{2}.$$

2. Since a_n is convergent it is a Cauchy sequence in $\mathbb{R}$. Note that a_n are rational numbers because each is the summation of two rational numbers. Therefore, it is a Cauchy sequence in $\mathbb{Q}$.
As it is shown in part (i), the sequence a_n converges to $\sqrt{2}$ which is not a member of $\mathbb{Q}$. Therefore, a_n cannot converge in $\mathbb{Q}$ and $\{a_n\}$ is not convergent in $\mathbb{Q}$. Thus, $\mathbb{Q}$ is not complete.

Problem 6 (Properties of DFT)**.**

Recall the DFT analysis and synthesis equations :

$$\text{Analysis equation : } X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$\text{Synthesis equation : } x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

Using the definitions we can now check the properties :

1) Linearity :

$$\begin{aligned}
z[n] &= \alpha x[n] + \beta y[n] \\
Z[k] &= \sum_{n=0}^{N-1} z[n] e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} (\alpha x[n] + \beta y[n]) e^{-j\frac{2\pi}{N}kn} \\
&= \alpha \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} + \beta \sum_{n=0}^{N-1} y[n] e^{-j\frac{2\pi}{N}kn} \\
&= \alpha X[k] + \beta Y[k].
\end{aligned}$$

2) Circular Shift :
Note that by taking mod N, we are only interested with shifts in the interval $0 \leq m \leq N-1$.

$$\begin{aligned}
z[n] &= x[(n-m) \bmod N] \\
Z[k] &= \sum_{n=0}^{N-1} z[n] e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} x[\underbrace{(n-m)}_{l} \bmod N] e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{l=0}^{N-1} x[l] e^{-j\frac{2\pi}{N}k((l+m) \bmod N)} \\
&= e^{-j\frac{2\pi}{N}km} \sum_{l=0}^{N-1} x[l] e^{-j\frac{2\pi}{N}kl} \qquad \text{since } e^{-j\frac{2\pi}{N}kn} \text{ is periodic with period } N \text{ in both } k, n \\
&= e^{-j\frac{2\pi}{N}km} X[k].
\end{aligned}$$

3) Duality

$$\begin{aligned}
z[n] &= X[n] \\
Z[k] &= \sum_{n=0}^{N-1} z[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} X[n] e^{-j\frac{2\pi}{N}kn} \\
Z[(-k) \bmod N] &= \sum_{n=0}^{N-1} X[n] e^{j\frac{2\pi}{N}kn} = N x[k] \\
Z[k] &= N x[-k \bmod N].
\end{aligned}$$

4) Symmetries

(i) $z[n] = x^*[n]$

$$\begin{aligned}
Z[k] &= \sum_{n=0}^{N-1} z[n]e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} x^*[n]e^{-j\frac{2\pi}{N}kn} \\
&= \left(\sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi}{N}kn}\right)^* \\
&= X^*[-k \bmod N]
\end{aligned}$$

(ii) $x_{ep}[n] = \frac{1}{2}\{x[n] + x^*[(-n) \bmod N]\}$

$$\begin{aligned}
X_{ep}[k] &= \frac{1}{2}\{\text{DFT}\{x[n]\} + \text{DFT}\{x^*[(-n) \bmod N]\}\} \\
&= \frac{1}{2}\{X[k] + X^*[k]\} \\
&= \text{Re}\,\{X[k]\}
\end{aligned}$$

(iii) $x_{op}[n] = \frac{1}{2}\{x[n] - x^*[(-n) \bmod N]\}$

$$\begin{aligned}
X_{ep}[k] &= \frac{1}{2}\{\text{DFT}\{x[n]\} - \text{DFT}\{x^*[(-n) \bmod N]\}\} \\
&= \frac{1}{2}\{X[k] - X^*[k]\} \\
&= j\,\text{Im}\,\{X[k]\}
\end{aligned}$$

5) <u>Cyclic convolution</u>

$$\begin{aligned}
z[n] &= \sum_{n=0}^{N-1} x[m]y[(n-m) \bmod N] \\
Z[k] &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x[m]y[(n-m) \bmod N]e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{m=0}^{N-1} x[m]\sum_{n=0}^{N-1} y[(n-m) \bmod N]e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{m=0}^{N-1} x[m]Y[k]e^{-j\frac{2\pi}{N}km} \\
&= Y[k]\sum_{m=0}^{N-1} x[m]e^{-j\frac{2\pi}{N}km} \\
&= Y[k]X[k].
\end{aligned}$$