ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 2 — Signal Processing for Communications

Solution 1 — February 21, 2011, INF213 - 10:15-12:00

Problem 1 (Continuity of a function)**.** Let a function $f(x)$ be continuous at x_0 if $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ s.t. if $|x - x_0| < \delta(\epsilon)$ then $|f(x) - f(x_0)| < \epsilon$, which means $\lim_{x \to x_0} f(x) = f(x_0)$.

(a) For $x \neq x_0$, the continuity of $\sin(\pi x)$ and πx implies that $\frac{\sin(\pi x)}{\pi x}$ is continuous. We can easily check continuity of $\sin(\pi x)$ by the definition.
For $x = x_0$, we have that $sinc(0) = 1$. Let us check the limit, which is an indeterminate form, $\frac{0}{0}$.

$$\lim_{x \to 0} \frac{\sin(\pi x)}{\pi x} = \lim_{x \to 0} \pi \frac{\cos(\pi x)}{\pi} = 1$$

The first equality comes from the l'Hospital rule.
Since $\lim_{x \to 0} sinc(x) = sinc(0)$, the function is continuous at $x = 0$. Then the function is continuous everywhere.

(b) Nowhere continuous means that it is not continuous at x_0 for all $x_0 \in \mathbb{R}$, i.e, $\exists \epsilon > 0$ s.t. $\forall \delta > 0$, there exists x such that if $|x - x_0| < \delta$, $|f(x) - f(x_0)| \not< \epsilon$.
We use the fact that any open intervals in $\mathbb{R}$ contain both rational and irrational numbers, i.e. there is an irrational number between any two rational and vice versa.
For $x_0 \in \mathbb{Q} : \epsilon = 0.5$, $\forall \delta > 0$, $\exists x \in \mathbb{R} \setminus \mathbb{Q}$ s.t. if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| = 1 \not< 0.5 \Rightarrow$ not continuous at $x_0 \in \mathbb{Q}$.
For $x_0 \in \mathbb{R} \setminus \mathbb{Q} : \epsilon = 0.5$, $\forall \delta > 0$, $\exists x \in \mathbb{R}$ s.t. if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| = 1 \not< 0.5 \Rightarrow$ not continuous at $x_0 \in \mathbb{Q}$. Then the function is nowhere continuous.

(c) Let $y_0 = g(x_0)$ and $h = f \circ g$.
Continuity of $f(.)$ implies that $\forall \epsilon > 0$, $\exists \delta_1 > 0$ such that $|y - y_0| < \delta_2 \Rightarrow |f(y) - f(y_0)| < \epsilon$. Then because of continuity of $g(.)$, for a given $\delta_1 > 0$, $\exists \delta > 0$ such that $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| = |y - y_0| < \delta_1$.
Therefore, by putting all together, we can say that

$\forall \epsilon > 0$, $\exists \delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(g(x)) - f(g(x_0))| = |h(x) - h(x_0)| < \epsilon$.

Problem 2 (Convergence of infinite series)**.** Let $S_n = \sum_{i=1}^{n} a_i$ where $a_i \in \mathbb{R}$. We say that the series $\sum_{i=1}^{n} a_i$ is *convergent* and has sum S, if $\lim_{n \to \infty} S_n = S$, i.e, for every $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that

$$|S - S_n| < \epsilon, \forall n > N.$$

(a) Assume that $\lim_{n \to \infty} S_n = S < \infty$.Then,

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1}) = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = S - S = 0.$$

Note that $\lim_{n \to \infty}(S_n - S_{n-1}) = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1}$ since their limit, S, is finite. For divergence series, this equality does not hold.

The other approach is using Cauchy convergence criterion : $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|\sum_{i=m}^{n} a_i| < \epsilon$ for $\forall m \geq n > N$. By taking $m = n + 1$, the desired result is obtained.

(b) We have that $s(x) = \frac{1}{x^p}$ is a decreasing positive function for all $x \in \mathbb{R}^+$. We want to prove that

$$\sum_{i=2}^{n} \frac{1}{i^p} \leq \int_1^n s(x)dx \leq \sum_{i=1}^{n-1} \frac{1}{i^p}. \tag{1}$$

Let $\int_1^n s(x)dx = \sum_{k=1}^{n-1} \int_k^{k+1} s(x)dx$. Since $s(x)$ is decreasing, $s(k) \geq s(x) \geq s(k+1)$ for all $x \in [k, k+1]$. Hence,

$$s(k+1) = \int_k^{k+1} s(k+1)dx \leq \int_k^{k+1} s(x)dx \leq \int_k^{k+1} s(k)dx = s(k).$$

Thus,

$$\sum_{i=2}^{n} \frac{1}{i^p} = \sum_{k=1}^{n-1} s(k+1) \leq \sum_{k=1}^{n-1} \int_k^{k+1} s(x)dx \leq \sum_{k=1}^{n-1} s(k) = \sum_{i=1}^{n-1} \frac{1}{i^p}.$$

Therefore, if $\lim_{n\to\infty} \int_1^n s(x)dx$ tends to infinity then the right inequality of (1) implies that $\sum_{i=2}^{\infty} \frac{1}{i^p}$ is divergent. On the other hand, if $\lim_{n\to\infty} \int_1^n s(x)dx$ is bounded, the left inequality of (1) implies that $\sum_{i=2}^{\infty} \frac{1}{i^p}$ is bounded and then it is convergent (since its elements are positive).

$$\sum_{i=1}^{\infty} s(n) \text{ converges} \Leftrightarrow \int_1^{\infty} s(x)dx \text{ converges.}$$

The evaluation of the integral gives

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} -px^{-p+1} \mid_1^{\infty} < \infty & \text{if } p > 1 \\ \ln x = \infty & \text{if } p = 1 \\ -px^{-p+1} \mid_1^{\infty} = \infty & \text{if } p < 1 \end{cases}$$

Then we conclude that the series converges only for $p > 1$.

(c) (i) $\sum_{n=1}^{\infty} x^n$ (geometric series). Let us first compute the sum up to N

$$\sum_{n=1}^{N} x^n = (1 + x + x^2 + \ldots + x^N) = \frac{1 - x^{N+1}}{1 - x},$$

where the second equality coming from the following factorization

$$(1 + x + x^2 + \ldots + x^N) \cdot (1 - x) = (1 + x + \ldots + x^N) - (x + x^2 + \ldots + x^{N+1}).$$

As N goes to infinity,

$$\lim_{N\to\infty} \sum_{n=1}^{N} x^n = \lim_{N\to\infty} \frac{1 - x^{N+1}}{1 - x} = \begin{cases} \frac{1}{1-x} & \text{if } |x| < 1 \\ \infty & \text{otherwise} \end{cases}$$

The condition on x for convergence is then $|x| < 1$.

(ii) In order to study the convergence of $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$, we first need to observe the following

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e}. \text{ Therefore, } \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2} = \frac{1}{e^n}.$$

Furthermore, let us recall the ratio test for convergence. Assume that for all n, $a_n > 0$. Suppose that there exists r such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

If $r < 1$, then the series converges. If $r > 1$, then the series diverges. If $r = 1$, the series may converge or diverge.

Thus

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{n+1}{n+2}\right)^{(n+1)^2}}{\left(\frac{n}{n+1}\right)^{n^2}} = \frac{\frac{1}{e^{n+1}}}{\frac{1}{e^n}} = \frac{1}{e} < 1.$$

Hence the series is convergent.

(iii) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$. By applying the bound $(n)! \leq n^{n+1}e^{-n}$, we get

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} \leq \sum_{n=1}^{\infty} \frac{n^{n+1}e^{-n}}{n^n} = \sum_{n=1}^{\infty} \cdot \frac{n}{e^n}$$

By using the ratio test, we can easily check that the upperbounded series is convergent and then $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent.

(iv) First note that $\frac{1}{i} - \frac{1}{i+1} \geq \frac{1}{j} - \frac{1}{j+1}$ for $j \geq i$. Then,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} &= -1 + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{4} - \frac{1}{5}) + (\frac{1}{6} - \frac{1}{7}) + \cdots \\
&< -1 + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + \cdots \\
&= -1 + \sum_{i=2}^{\infty} \frac{1}{i} - \frac{1}{i+1} = -1 + \frac{1}{2} = \frac{-1}{2}.
\end{aligned}$$

Since the summation of two subsequent elements ,$(\frac{1}{i} - \frac{1}{i+1})$, is positive, for odd N, the $\sum_{n=1}^{N} \frac{(-1)^n}{n}$ is increasing and bounded from above by $\frac{-1}{2}$. Therefore, the series is convergent.

(d) Two alternatives of proofs are presented

Alternative 1 Let us define $a_i^+ = \max\{a_i, 0\}$ and $a_i^- = \max\{-a_i, 0\}$. Then

$$|a_i| = |a_i^+| + |a_i^-|, \text{ and } a_i = |a_i^+| - |a_i^-|.$$

The series can be expressed using a_i^+ and a_i^-

$$\sum_{i=1}^{\infty} |a_i| = \sum_{i=1}^{\infty} |a_i^+| + |a_i^-| < \infty \Rightarrow \sum_{i=1}^{\infty} |a_i^+| < \infty \quad \text{and} \sum_{i=1}^{\infty} |a_i^-| < \infty.$$

Then $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} |a_i^+| - \sum_{i=1}^{\infty} |a_i^-| < \infty$.

Alternative 2 We can write

$$\left| \sum_{i=n}^{m} a_i \right| \leq \sum i = n^m |a_i| < \epsilon \text{ by assumption.}$$

Hence it is a Cauchy sequence and so it converges.

Problem 3 (Sums)**.** In parts (i), (ii) and (iv) we can directly see that the sums are finite.

(i)

$$\begin{aligned}\sum_{k=i}^{n} x^k &= \sum_{k=0}^{n} x^k - \sum_{k=0}^{i-1} x^k \\ &= \frac{1-x^{n+1}}{1-x} - \frac{1-x^i}{1-x} \\ &= \frac{x^i - x^{n+1}}{1-x}\end{aligned}$$

(ii)

$$\begin{aligned}\sum_{k=1}^{n} kx^k &= x\sum_{k=1}^{n} kx^{k-1} \\ &= x\sum_{k=1}^{n} \frac{dx^k}{dx} \\ &= x\frac{d}{dx}\left(\sum_{k=1}^{n} x^k\right) \text{ by linearity of differentiation} \\ &= x\frac{d}{dx}\left(\frac{1-x^{n+1}}{1-x} - 1\right) \\ &= x\frac{-(n+1)x^n(1-x) - (1-x^{n+1})(-1)}{(1-x)^2} \\ &= \frac{x-(n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}\end{aligned}$$

(iii) $\sum_{n=1}^{\infty}\left(\frac{\sqrt{3}}{2} + \frac{1}{j2}\right)^n$

Let $x = \frac{\sqrt{3}}{2} + \frac{1}{j2}$. In problem 2 part c), you found the condition for convergence on $\sum_{n=1}^{\infty} x^n$ as $|x| < 1$. Note that here $|x| = |\frac{\sqrt{3}}{2} + \frac{1}{j2}| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1 \not< 1$.
Hence this series diverges and the sum is infinite.

(iv)

$$\begin{aligned}\sum_{k=1}^{n} \sin(2\pi\frac{k}{N}) &= \sum_{k=1}^{n} \frac{e^{j2\pi\frac{k}{N}} - e^{-j2\pi\frac{k}{N}}}{2j} \\ &= \frac{1}{2j}\left(\frac{1-e^{j2\pi\frac{n+1}{N}}}{1-e^{j\frac{2\pi}{N}}} - 1 - \frac{1-e^{-j2\pi\frac{n+1}{N}}}{1-e^{-j\frac{2\pi}{N}}} + 1\right) \\ &= \frac{1}{2j}\frac{1-e^{-j\frac{2\pi}{N}} - e^{j2\pi\frac{n+1}{N}} + e^{j2\pi\frac{n}{N}} - 1 + e^{j\frac{2\pi}{N}} + e^{-j2\pi\frac{n+1}{N}} - e^{-j2\pi\frac{n}{N}}}{\left(1-e^{j\frac{2\pi}{N}}\right)\left(1-e^{-j\frac{2\pi}{N}}\right)} \\ &= \frac{\sin(\frac{2\pi}{N}) + \sin(2\pi\frac{n}{N}) - \sin(2\pi\frac{n+1}{N})}{2-2\cos(\frac{2\pi}{N})}\end{aligned}$$

Problem 4 (Inner Product Properties). (a)

$$\begin{aligned}
\|x+y\|^2 &= <x+y, x+y> \\
&= <x,x> + <x,y> + <y,x> + <y,y> \\
&= \|x\|^2 + <x,y> + \overline{<x,y>} + \|y\|^2 \\
&= \|x\|^2 + 2\,\mathrm{Re}\{<x,y>\} + \|y\|^2
\end{aligned}$$

When $E = \mathbb{R}^2$, using the definition of the inner product on $\mathbb{R}^2$, we have

$$<x,y> = \|x\|\|y\|\cos\theta$$

where θ is the angle between vectors x and y. We see that

$$\forall x, y, \text{ s.t. } x \perp y \Longleftrightarrow <x,y> = 0$$

Plugging this into the expression, we recover the famous Pythagorean Formula

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

(b) Using the previous expression, we have

$$\begin{aligned}
\|x+y\|^2 &= \|x\|^2 + 2\,\mathrm{Re}\{<x,y>\} + \|y\|^2 \\
\|x-y\|^2 &= \|x\|^2 - 2\,\mathrm{Re}\{<x,y>\} + \|y\|^2
\end{aligned}$$

Adding the two components, we obtain

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{2}$$

(c) Note first that

$$\begin{aligned}
\|\alpha x + \beta y\| &= <\alpha x + \beta y, \alpha x + \beta y> \\
&= |\alpha|^2 <x,x> + \alpha\bar{\beta} <x,y> + \beta\bar{\alpha} <y,x> + |\beta|^2 <y,y>
\end{aligned}$$

Hence we get,

$$\begin{aligned}
&\frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-y\|^2\} \\
=&\frac{1}{4}\{<x,x> + <x,y> + <y,x> + <y,y> \\
&- <x,x> + <x,y> + <y,x> - <y,y> \\
&+ i <x,x> + <x,y> - <y,x> - i <y,y> \\
&- i <x,x> + <x,y> - <y,x> + i <y,y>\} \\
=&\frac{1}{4} <x,y>
\end{aligned}$$

as required.

Now we check that the polarization identity does indeed satisfy the properties of an inner product.

1)

$$\begin{aligned} < x, x > &= \frac{1}{4}\{4\|x\|^2 + i|1+i|^2\|x\|^2 - i|1-i|^2\|x\|^2\} \\ &= \|x\|^2 \geq 0 \text{ with equality iff } x = 0. \end{aligned}$$

2)

$$\begin{aligned} \overline{< y, x >} &= \frac{1}{4}\{\|y+x\|^2 - \|y-x\|^2 - i\|y+ix\|^2 + i\|y-ix\|^2\} \\ &= \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2 - i|i|^2\|x-iy\|^2 + i|i|^2\|x+iy\|^2\} \\ &= \frac{1}{4}\{\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 + i\|x+iy\|^2\} \\ &= < x, y > . \end{aligned}$$

3)

$$< x+y, z > = \frac{1}{4}\{\|x+y+z\|^2 - \|x+y-z\|^2 + i\|x+y+iz\|^2 - i\|x+y-iz\|^2\}$$

Let us first compute

$$\begin{aligned} \|x+y+z\|^2 - \|x+y-z\|^2 &= \left\|\left(x+\frac{z}{2}\right) + \left(y+\frac{z}{2}\right)\right\|^2 - \left\|\left(x-\frac{z}{2}\right) + \left(y-\frac{z}{2}\right)\right\|^2 \\ &= 2\|x+\frac{z}{2}\|^2 + 2\|y+\frac{z}{2}\|^2 - \|x-y\|^2 \\ &- \|x-\frac{z}{2}\|^2 - 2\|y-\frac{z}{2}\|^2 + \|x-y\|^2 \\ &= 2\{\|x+\frac{z}{2}\|^2 - \|x-\frac{z}{2}\|^2 + \|y+\frac{z}{2}\|^2 - \|y-\frac{z}{2}\|^2\} \end{aligned}$$

Similarly when z is replaced by iz, we have

$$i\|x+y+iz\|^2 - i\|x+y-iz\|^2 = 2\{i\|x+i\frac{z}{2}\|^2 - i\|x-i\frac{z}{2}\|^2 + i\|y+i\frac{z}{2}\|^2 - i\|y-i\frac{z}{2}\|^2\}$$

Therefore,

$$< x+y, z >= 2 < x, \frac{z}{2} > +2 < y, \frac{z}{2} >$$

As we have already assumed $< \alpha x, y >= \alpha < x, y >$ holds and we get the additivity in the first component property

$$< x+y, z >=< x, z > + < y, z > .$$