Signal Processing for Communications

EPFL Spring Semester 2009
April 21, 2009

Solutions Midterm

You have 2 hours. We do not necessarily expect that you finish all the problems. Don't lose time on any particular problem, but try to get as many points as you can.

You are allowed to have one piece of A4 paper as crypt sheet. Please, no calculators, cell phones, neighbors, books, notes, formula collections, ...

Good luck ! !

Name : ______________________________

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Total	/100

<u>Problem I</u> (25 points)

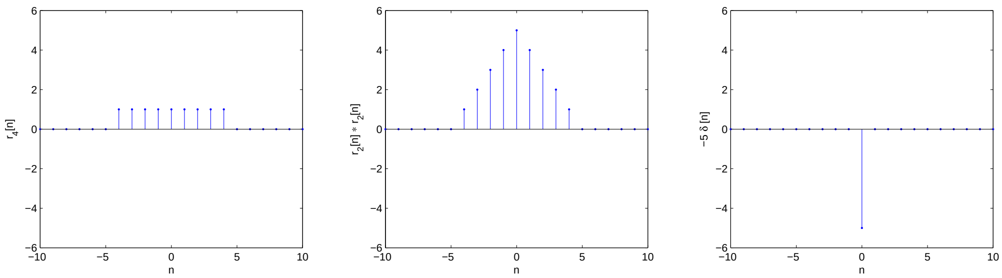
a) We know that the inverse DTFT of $R(T)$ is given by

$$r_T[n] = \begin{cases} 1 & \text{if } |n| \leq T \\ 0 & \text{otherwise} \end{cases}.$$

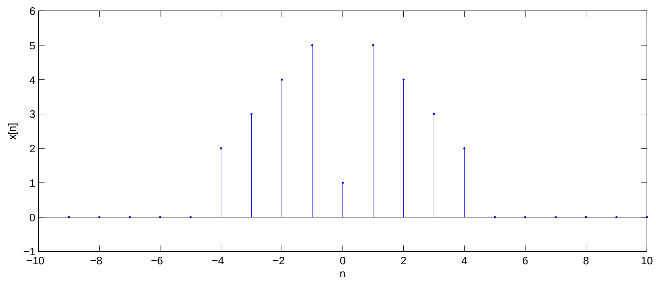
We also know that $\delta[n] \overset{\text{DTFT}}{\Longleftrightarrow} 1$. Therefore, using the convolution-multiplication duality we get

$$x[n] = r_4[n] + r_2[n] * r_2[n] - 5\delta[n].$$

$r_4[n]$, $r_2[n] * r_2[n]$ and $-5\delta[n]$ are plotted below.



The sum of the above signals, $x[n]$, is plotted in the following figure :



b) Note that the $x[n]$ we found in part 1 is real and symmetric, i.e., $x[n] = x[-n]$. Therefore, $x[n] + x[-n] = 2x[n] \overset{\text{DTFT}}{\Longleftrightarrow} 2X(e^{j\omega})$. Therefore, $|W(e^{j\omega})| = 2|X(e^{j\omega})|$ (see figure below).

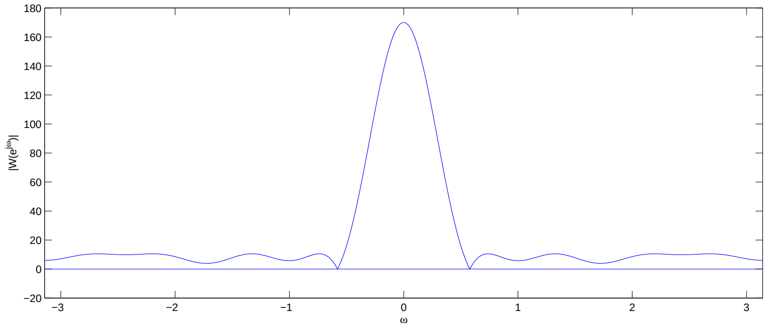
c) Keeping in mind the symmetry of the signal $x[n]$, we have $x[-n+1] = x[n-1]$. Therefore,

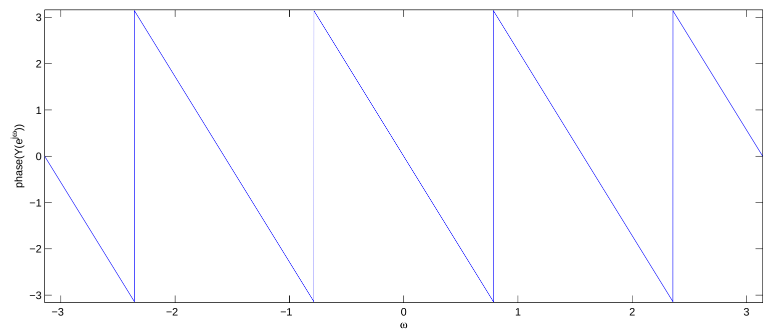
$$y[n] = x[n-1] * h[n] = x[n-1] * \delta[n-3] = x[n-4].$$

Hence we have

$$y[n] = x[n-4] \overset{\text{DTFT}}{\Longleftrightarrow} e^{-j4\omega} X(e^{j\omega}) = Y(e^{j\omega}).$$

Note that $X(e^{j\omega})$ is real-valued. Therefore, the phase of $Y(e^{j\omega})$ is -4ω. This is plotted in the figure below.





<u>Problem II</u> (25 points)

a) [5 pts] The z-transform of the impulse response is :

$$H(z) = \frac{1 + z^{-1}}{(1 - z^{-1})(1 - 2z^{-1})(1 + 2z^{-1})}$$

Thus the poles are at $z = 1$,$z = \pm 2$. There is a zero at $z = -1$ and a zero of multiplicity 2 at $z = 0$.

b) [10 pts] So in general we can have three systems. None of them are stable (because of the pole which is exactly on the unit circle). The system whose ROC corresponds to the outside of the rightmost pole is casual (also the limit of $H(z)$ when z goes to infinity is one).

c) [10 pts] For this system the impulse response can be obtained as follows. Since all the poles are simple we know that the partial fraction has the form

$$H(z) = \frac{a}{1 - z^{-1}} + \frac{b}{1 - 2z^{-1}} + \frac{c}{1 + 2z^{-1}}.$$

There are now several equivalent ways of determining the coefficients. The first is to bring the previous expression to a common denominator and then to compare coefficients. We obtain

$$a(1 - 4z^{-2}) + b(1 - 2z^{-2} + z^{-1}) + c(1 + 2z^{-2} - 3z^{-1}) = 1 + z^{-1}.$$

This results in the following set of equations :

$$\begin{aligned} a + b + c &= 1, \\ -4a - 2b + 2c &= 0, \\ b - 3c &= 1. \end{aligned}$$

Solving the above we obtain $a = -\frac{2}{3}$, $b = \frac{3}{2}$, $c = \frac{1}{6}$. Thus we have :

$$h[n] = (-\frac{2}{3} + \frac{3}{2}2^n + \frac{1}{6}(-2)^n)u[n].$$

Alternatively, we know that we can get the coefficients directly as

$$
\begin{aligned}
a &= H(z)(1 - z^{-1}) \mid_{z=1} = -\frac{2}{3}, \\
b &= H(z)(1 - 2z^{-1}) \mid_{z=2} = \frac{3}{2}, \\
c &= H(z)(1 + 2z^{-1}) \mid_{z=-2} = \frac{1}{6}.
\end{aligned}
$$

Problem III **(25 points)**

a) [7 pts] The matrix W is the usual (M-Point) DFT matrix.

b) [8 pts] Because of the reconstruction formula we have : $\underline{X} = \frac{1}{M} W^H \underline{X}$, so using part (a) we have : $\underline{X} = \frac{1}{M} W^H W \underline{X}$. Thus we have : $\frac{1}{M} W^H W = I_{M\times M}$ and the result follows.

c) [10 pts] Similarly to part (a), taking t-times M-point DFT corresponds to multiplying the matrix W, t times. So if we define the vectors $\underline{X}_1$ and $\underline{X}_2$ as $\underline{X}_1 = [X_1[0], X_1[1], \ldots, X_1[M-1]]^T$ and $\underline{X}_2 = [X_2[0], X_2[1], \ldots, X_2[M-1]]^T$, we would obtain : $\underline{X}_1 = W^t \underline{X}_1$ and $\underline{X}_2 = W^t \underline{X}_2$. Thus :

$$(\underline{X}_1)^H \underline{X}_2 = (W^H W)^t (\underline{X}_1)^H \underline{X}_2$$

Thus according to part (b) we have

$$(\underline{X}_1)^H \underline{X}_2 = M^t (\underline{X}_1)^H \underline{X}_2$$

And the rest is clear.

Problem IV (25 points)

a) [8 pts] The system represents a "fractional" delay (see the last point). Hence, $y[n] = \sin((n - d)\omega_0 + \phi_0)$, i.e., $y[n]$ is "delayed" by d time units. The simplest way to compute $y[n]$ is to transform into the Fourier domain, to multiply, and to transform back.

b) [7 pts] We have $h[n] = \delta[n-d]$. Since this impulse response is absolutely summable, the system is BIBO stable. If $d \geq 0$ then the system is causal, otherwise it is not.

c) [10 pts] If d is not an integer then we get by direct integration

$$\begin{aligned} h[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\omega d} e^{j\omega n} d\omega \\ &= \frac{1}{2j\pi(n-d)} e^{j\omega(n-d)} \Big|_{-\pi}^{\pi} \\ &= \frac{e^{j\pi(n-d)} - e^{-j\pi(n-d)}}{2j\pi(n-d)} \\ &= \operatorname{sinc}(n-d). \end{aligned}$$

In this case the impulse response is not absolutely summable, so the system is not BIBO stable. The system is never causal.