Problem 1 (This Only Looks Complex)

i) Take \( X = e^{-\frac{j2\pi}{N}m} \), then :

\[
S = \sum_{n=0}^{N-1} e^{-\frac{j2\pi}{N}mn} = \sum_{n=0}^{N-1} X^n = 1 + X + \cdots + X^{N-1}.
\]

If \( X = 1 \), then \( S = N \). Otherwise, it is a geometric series and

\[
S = \frac{1 - X^N}{1 - X} = 0 \quad \text{since} \quad X^N = (e^{-\frac{j2\pi}{N}m})^N = e^{-j2\pi m} = 1.
\]

ii) \( X^N - 1 = (X - 1)(1 + X + \cdots + X^{N-1}) = 0. \)

From the previous part, we know that \( X = e^{-\frac{j2\pi}{N}m} \) for \( m = 0, 1, \ldots, N - 1 \) are the roots of \( X^N - 1 = 0. \)

iii) Note that

\[
\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \text{and} \quad e^{j\theta} = \cos \theta + j \sin \theta \Rightarrow \cos \theta = \text{Re}\{e^{j\theta}\}.
\]

Therefore,

\[
\sum_{n=0}^{N-1} \cos^2 \left( \frac{2\pi}{N} kn \right) = \sum_{n=0}^{N-1} \frac{1 + \cos \left( \frac{2\pi}{N} 2kn \right)}{2} = \frac{N}{2} + \frac{1}{2} \sum_{n=0}^{N-1} \cos \left( \frac{2\pi}{N} 2kn \right).
\]

But,

\[
\sum_{n=0}^{N-1} \cos \left( \frac{2\pi}{N} 2kn \right) = \sum_{n=0}^{N-1} \text{Re}\{e^{j \frac{2\pi}{N} 2kn}\} = \text{Re} \left\{ \sum_{n=0}^{N-1} e^{j \frac{2\pi}{N} kn} \right\}
\]

Since for \( 2k \neq iN, i \in \mathbb{Z}, \sum_{n=0}^{N-1} e^{j \frac{2\pi}{N} kn} = 0 \), we conclude that

\[
\sum_{n=0}^{N-1} \cos \left( \frac{4\pi}{N} kn \right) = \text{Re}\{0\} = 0.
\]

Finally :

\[
\sum_{n=0}^{N-1} \cos^2 \left( \frac{2\pi}{N} kn \right) = \frac{N}{2}.
\]

Problem 2 (Black Box)
i) A complex signal $c[n]$ can be written as a summation of two real signals: real part and imaginary part which is multiplied by $j$. It means that

$$c[n] = x[n] + jy[n]$$

where $x[n] = \frac{c[n] + c^*[n]}{2}$ is the real part of the signal and $y[n] = \frac{c[n] - c^*[n]}{2j}$ is the imaginary part of the signal. Note that $c^*[n]$ is the complex conjugate of $c[n]$.

Since DFT operator is a linear operator, i.e. $DFT(\alpha a[n] + \beta b[n]) = \alpha DFT(a[n]) + \beta DFT(b[n])$, we can conclude that

$$DFT(c[n]) = DFT(x[n]) + j DFT(y[n]) = F^k_N(x[n]) + j F^k_N(y[n])$$

$$\Rightarrow G_N^k(c[n]) = F_N^k\left(\frac{c[n] + c^*[n]}{2}\right) + j F_N^k\left(\frac{c[n] - c^*[n]}{2}\right)$$

We could also reach to the above equation by explicitly using the DFT expansion.

ii) By using Problem (6.3) of homework 2, if $C[k]$ is the $N$-point DFT of the signal $c[n]$, then

$$c[n] \xrightarrow{DFT} Nc[(-k)modN]$$

or equivalently,

$$C[(-n)modN] \xrightarrow{DFT} NC[k].$$

Therefore,

$$c[k] = \frac{1}{N} G_N^k(C[(-n)modN]).$$

iii) Since $G_N^k(\cdot)$ functionals can operate on signals with length at most $N$, we should divide $x[n]$ into two signals with length $N$. One way is to divide $x[n]$ to odd and even components:

$$x_e[\ell] = x[2\ell] \text{ and } x_o[\ell] = x[2\ell + 1], \text{ for } \ell = 0, 1, \ldots, N - 1.$$ 

Then, for $K = 0, 1, \ldots, 2N - 1$,

$$X[K] = \sum_{n=0}^{2N-1} x[n] e^{-j \frac{2\pi}{N} Kn} = \sum_{n=0}^{N-1} x[2\ell] e^{-j \frac{2\pi}{N} (2\ell)} + \sum_{n=0}^{N-1} x[2\ell + 1] e^{-j \frac{2\pi}{N} (2\ell + 1)}$$

$$= \sum_{\ell=0}^{N-1} x_e[\ell] e^{-j \frac{2\pi K}{N} \ell} + e^{-j \frac{\pi K}{N}} \sum_{\ell=0}^{N-1} x_o[\ell] e^{-j \frac{2\pi K}{N} \ell} = X_e[K] + e^{-j \frac{\pi K}{N}} X_o[K].$$

Note that the DFT of $x_e[n]$ and $x_o[n]$ is $N$-point DFT and therefore $X_e[K+\mathbb{N}] = X_e[K]$ for $K = 0, 1, \ldots, N - 1$. Hence,

$$X[K] = \begin{cases} 
X_e[K] + e^{-j \frac{\pi K}{N}} X_o[K] & 0 \leq K < N \\
X_e[K-N] + e^{-j \frac{\pi K}{N}} X_o[K-N] & N \leq K < 2N 
\end{cases}$$

or

$$X[K] = \begin{cases} 
G_N^K x_e[n] + e^{-j \frac{\pi K}{N}} G_N^K x_o[n] & 0 \leq K < N \\
G_N^{K-N} x_e[n] + e^{-j \frac{\pi K}{N}} G_N^{K-N} x_o[n] & N \leq K < 2N 
\end{cases}$$

**Problem 3**
i) We show that it is not summable. On the contrary, assume that it is summable. Then for every \( \varepsilon > 0 \), there exists a finite set \( J_\varepsilon \subset \mathbb{N} \) such that for every finite set \( K \subset \mathbb{N} \),
\[
|\sum_{n \in K} a_n| < \varepsilon \quad \text{if} \quad K \cap J_\varepsilon = \emptyset.
\]

We show that it is not possible to have such \( J_\varepsilon \). As \( J_\varepsilon \) is finite set in \( \mathbb{N} \), then it has a maximum member. Let \( M = \max J_\varepsilon \) then \( J_\varepsilon \subset \{1, 2, \ldots, M\} \). Define \( K^{(L)} = \{2M, 2(M+1), \ldots, 2(M + L)\} \). \( K^{(L)} \cap J_\varepsilon = \emptyset \) for all \( L \) values but :
\[
\sum_{n \in K^{(L)}} a_n = \frac{1}{2} \sum_{n=M}^{L+M} \frac{1}{n} > \frac{1}{2} \int_M^{M+L} \frac{1}{x} dx + \frac{1}{2M} = \frac{1}{2} \ln \frac{M + L}{M} + \frac{1}{2M}.
\]

The inequality could be easily verified similar to what we did in Problem (2.b) in homework 1. Hence,
\[
\sum_{n \in K^{(L)}} a_n > \frac{1}{2} \ln(1 + \frac{L}{M}) + \frac{1}{2M}.
\]

For arbitrary large \( L \), \( |\sum_{n \in K^{(L)}} a_n| \) could take any large value. Therefore, there exists many \( K \subset \mathbb{N} \) such that \( K \cap J_\varepsilon = \emptyset \) and \( \sum_{n \in K} a_n > \varepsilon \) for every finite \( J_\varepsilon \) and \( \varepsilon > 0 \).

ii) \( \sum_{n=1}^{\infty} (-\frac{1}{n})^n \) is absolutely convergent if \( \sum_{n=1}^{\infty} \frac{1}{n} \) is convergent.

\( \sum_{n=1}^{\infty} \frac{1}{n} \) is harmonic series and in the Problem (2.b) of homework 1 we have shown that it is divergent. Hence, \( \sum_{n=1}^{\infty} (-\frac{1}{n})^n \) is not absolutely convergent. However, it is convergent.

iii) Let’s begin with the sufficient condition. If \( \sum_{n=1}^{\infty} a_n \) converge absolutely, then \( \{a_n\} \) is summable.

Consider \( S_n = \sum_{i=1}^{n} |a_i| \). Since \( S_n < \infty \) (it is bounded) and \( S_n \) is an increasing sequence, then it converges to a value, call it \( S \). In other words, for every \( \varepsilon > 0 \),
\[
\exists M(\varepsilon) \in \mathbb{N} \text{ such that } n \geq M(\varepsilon) : \left| \sum_{i=1}^{n} |a_i| - S \right| < \varepsilon
\]

For a given \( \varepsilon > 0 \), let \( n = M(\varepsilon) \). Thus,
\[
\left| \sum_{i=1}^{M(\varepsilon)} |a_i| - S \right| = \left| \sum_{i=1}^{M(\varepsilon)} |a_i| - \sum_{i=1}^{\infty} |a_i| \right| < \varepsilon
\]

But \( \sum_{i=1}^{M(\varepsilon)} |a_i| - \sum_{i=M(\varepsilon)+1}^{\infty} |a_i| = \sum_{i=M(\varepsilon)+1}^{\infty} |a_i| \). Therefore, \( \sum_{i=M(\varepsilon)+1}^{\infty} |a_i| < \varepsilon \). Let \( J_\varepsilon = \{1, 2, \ldots, M(\varepsilon)\} \), then every finite set \( K \subset \mathbb{N} \) such that \( K \cap J_\varepsilon = \emptyset \) is a subset of \( \{M(\varepsilon) + 1, M(\varepsilon) + 2, \ldots\} \) and consequently,
\[
\sum_{i \in K} |a_i| < \sum_{i=M(\varepsilon)+1}^{\infty} |a_i| < \varepsilon.
\]

It means that \( \{a_i\} \) is summable.

Now, we prove the necessary condition. If \( \{a_n\} \) is summable, then \( \sum a_n \) converges absolutely.
Let us first assume that \( \{a_n\} \) is a real sequence. If \( \{a_n\} \) is summable, for a given \( \varepsilon > 0 \), there exists \( J_\varepsilon \) such that for every \( K \) finite set and \( K \cap J_\varepsilon = \emptyset : \left| \sum_{n \in K} a_n \right| < \varepsilon \).

Consider a given set \( K \). We split it into the finite set \( K^+ \) with positive elements and the finite set \( K^- \) with negative elements. Therefore, \( K^+ \cap J_\varepsilon = \emptyset \) and \( K^- \cap J_\varepsilon = \emptyset \). Then:

\[
\left| \sum_{n \in K^+} a_n \right| = \sum_{n \in K^+} |a_n| < \varepsilon \\
\Rightarrow \sum_{n \in K^+ \cup K^- = K} |a_n| < 2\varepsilon.
\]

\[
\left| \sum_{n \in K^-} a_n \right| = \sum_{n \in K^-} |a_n| < \varepsilon
\]

Therefore, for every finite set \( K \) such that \( K \cap J_\varepsilon = \emptyset \):

\[
\sum_{n \in K} |a_n| < 2\varepsilon.
\]

Since we can take \( K \) arbitrarily large, thus \( \sum_{n \in \mathbb{N}} |a_n| = \sum_{n \in J_\varepsilon} |a_n| + \sum_{n \notin J_\varepsilon} |a_n| \) is bounded and then it is absolutely convergent.

For the complex sequence, every complex sequence can be written as a summation of two (real and imaginary) sequences:

\[
a_n = x_n + jy_n, x_n, y_n \in \mathbb{R}
\]

where \( |x_n| < |a_n| \) and \( |y_n| < |a_n| \) and \( |a_n| < |x_n| + |y_n| \).

It can be easily verified that if \( \{a_n\} \) is summable then \( \{x_n\} \) and \( \{y_n\} \) are summable and consequently \( \sum x_n \) and \( \sum y_n \) are absolutely convergent. Hence, \( \sum |a_n| < \sum |x_n| + \sum |y_n| < \infty \) is absolutely convergent.

**Problem 4**

i) We are looking for the coefficients \( \{\alpha_0, \alpha_1, \ldots, \alpha_n\} \) such that:

\[
\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n = 0 \quad \forall x \in [0, 1]
\]

If \( \alpha_n \neq 0 \), then the above polynomial with degree \( n \) has at most \( n \) different solutions but it should be zero for all \( x \in [0, 1] \). Therefore, it is not possible unless all coefficients are equal to zero. Thus, \( \{1, x, \ldots, x^n\} \) are linearly independent.

ii) To find the orthonormal basis for a given set of vectors, we use Gram-Schmidt procedure. In the space of \( C[0, 1] \) define \( v_0 = 1, v_1 = x^1, \ldots, v_n = x^n \).

Then \( u_0(x) = \frac{v_0}{||v_0||} = 1, ||v_0|| = \sqrt{<1, 1>} = \sqrt{\int_0^1 1 \, dx} = 1 \)

\[
u_1(x) = \frac{v_1 - \langle v_1, u_0 \rangle u_0}{||v_1 - \langle v_1, u_0 \rangle u_0||} = \frac{x < 1, x >}{||x < 1, x >||}
\]

\[
<1, x> = \int_0^1 x \, dx = \frac{1}{2}, ||x - \frac{1}{2}|| = \sqrt{< x - 1/2, x - 1/2>} = \sqrt{\int_0^1 (x - \frac{1}{2})^2 \, dx} = \frac{1}{\sqrt{12}}
\]
Therefore, \( u_1(x) = \sqrt{12}(x - 1/2) \).

The other orthonormal elements of the basis can be made by the following recursion:

\[
u_i(x) = \frac{x^i - \sum_{j=0}^{i-1} x^j, u_i(x) > u_i(x)}{||x^i - \sum_{j=0}^{i-1} x^j, u_i(x) > u_i(x)||}.
\]

iii) By the projection theorem, \( v_p \in B \) is the projection of a vector \( v \) in the Hilbert subspace \( B \), if

\[||v - v_p|| = \inf_{w \in B} ||v - w||.\]

To find a polynomial with degree \( n \), \( P_n(x) \), which has the minimum total squared error with \( p(x) \), i.e. \( ||p(x) - P_n(x)||^2 \), we should look for the projection of \( p(x) \) in the Hilbert space of all polynomial functions of degree \( n \) with the norm \( || \cdot || \).

Assume that \( P_n(x) = \sum_{i=0}^n b_i u_i(x) \), then:

\[b_i = \langle P_n(x), u_i(x) \rangle = \langle p(x), u_i(x) \rangle = \int_0^1 p(x) u_i^*(x) dx.\]

iv) We should find the projection of \( \sin \frac{\pi}{2} x \) in the space of polynomials with degree 2. The orthonormal basis \( \{u_0(x), u_1(x), u_2(x)\} \) is equal to:

\[u_0(x) = 1, \quad u_1(x) = \sqrt{12}(x - 1/2), \quad u_2(x) = \frac{x^2 - < x^2, 1 > - < x^2, u_1(x) > u_1(x) - u_1(1)}{||x^2 - < x^2, 1 > - < x^2, u_1(x) > u_1(1)||} = 6\sqrt{5}(x^2 - x + 1/6).\]

Therefore,

\[b_0(x) = \langle \sin \frac{\pi}{2} x, 1 \rangle = \int_0^1 \sin \frac{\pi}{2} x dx = \frac{2}{\pi}, \]

\[b_1(x) = \langle \sin \frac{\pi}{2} x, u_1(x) \rangle = \sqrt{12} \frac{4}{\pi^2} - \sqrt{12} \frac{1}{\pi}, \]

\[b_2(x) = \langle \sin \frac{\pi}{2} x, u_2(x) \rangle = 6\sqrt{5} \left( \frac{4}{\pi^2} - \frac{16}{\pi^3} \right) + \frac{2\sqrt{5}}{\pi}.\]

The two last are concluded since:

\[\int_0^1 x \sin \frac{\pi}{2} x dx = \frac{2x}{\pi} \cos \frac{\pi}{2} x |^1_0 + \frac{2}{\pi} \int_0^1 \cos \frac{\pi}{2} x dx = \frac{4}{\pi^2} \sin \frac{\pi}{2} x |^1_0 = \frac{4}{\pi^2}, \]

\[\int_0^1 x^2 \sin \frac{\pi}{2} x dx = \frac{2x^2}{\pi} \cos \frac{\pi}{2} x |^1_0 + \frac{4}{\pi} \int_0^1 x \cos \frac{\pi}{2} x dx = \frac{8}{\pi^2} x \sin \frac{\pi}{2} x |^1_0 - \frac{8}{\pi^2} \int_0^1 \sin \frac{\pi}{2} x dx = \frac{8}{\pi^2} + \frac{16}{\pi^4} \cos \frac{\pi}{2} x |^1_0 = \frac{8}{\pi^2} + \frac{16}{\pi^4}.\]
We solved the two integrals by using integration by part. Finally,

\[ P_2(x) = b_0 + b_1 u_1(x) + b_2 u_2(x) = -0.024 + 1.878x - 0.834x^2. \]

If we proceed one degree more, the degree 3 approximated polynomial is:

\[ P_3(x) = P_2(x) + b_3 u_3(x) = -0.002 + 1.6134x - 0.1724x^2 - 0.4413x^3. \]

In figure 1, the plots of \( \sin \frac{\pi}{2}x \), \( P_2(x) \) and \( P_3(x) \) are depicted. We can see that \( P_3(x) \) is located very close to \( \sin \frac{\pi}{2}x \).