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Handout 8 Signal Processing for Communications
Solution 4 March 21, 2011, INF213 – 10:15am-12:00

Problem 1 (This Only Looks Complex)

i) Take $X = e^{-\frac{j2\pi}{N}m}$, then :

$$S = \sum_{n=0}^{N-1} e^{-\frac{j2\pi}{N}mn} = \sum_{n=0}^{N-1} X^n = 1 + X + \cdots + X^{N-1}.$$

If $X = 1$, then $S = N$. Otherwise, it is a geometric series and

$$S = \frac{1-X^N}{1-X} = 0 \qquad \text{since } X^N = (e^{-\frac{j2\pi}{N}m})^N = e^{-j2\pi m} = 1.$$

ii) $X^N - 1 = (X-1)(1 + X + \cdots + X^{N-1}) = 0$.
From the previous part, we know that $X = e^{-\frac{j2\pi}{N}m}$ for $m = 0, 1, \ldots, N-1$ are the roots of $X^N - 1 = 0$.

iii) Note that

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \text{and} \qquad e^{j\theta} = \cos\theta + j\sin\theta \Rightarrow \cos\theta = Re\{e^{j\theta}\}.$$

Therefore,

$$\sum_{n=0}^{N-1} \cos^2(\frac{2\pi}{N}kn) = \sum_{n=0}^{N-1} \frac{1+\cos(\frac{2\pi}{N}2kn)}{2} = \frac{N}{2} + \frac{1}{2}\sum_{n=0}^{N-1}\cos(\frac{2\pi}{N}2kn).$$

But,

$$\sum_{n=0}^{N-1}\cos(\frac{2\pi}{N}2kn) = \sum_{n=0}^{N-1} Re\{e^{j\frac{2\pi}{N}2kn}\} = Re\left\{\sum_{n=0}^{N-1} e^{j\frac{4\pi}{N}kn}\right\}$$

Since for $2k \neq iN, i \in \mathbb{Z}, \sum_{n=0}^{N-1} e^{j\frac{4\pi}{N}kn} = 0$, we conclude that

$$\sum_{n=0}^{N-1} \cos\frac{4\pi}{N}kn = Re\{0\} = 0.$$

Finally :

$$\sum_{n=0}^{N-1} \cos^2(\frac{2\pi}{N}kn) = \frac{N}{2}.$$

Problem 2 (Black Box)

i) A complex signal $c[n]$ can be written as a summation of two real signals : real part and imaginary part which is multiplied by j. It means that

$$c[n] = x[n] + jy[n]$$

where $x[n] = \frac{c[n]+c^*[n]}{2}$ is the real part of the signal and $y[n] = \frac{c[n]-c^*[n]}{2j}$ is the imaginary part of the signal. Note that $c^*[n]$ is the complex conjugate of $c[n]$.

Since DFT operator is a linear operator, i.e. $DFT(\alpha a[n] + \beta b[n]) = \alpha DFT(a[n]) + \beta DFT(b[n])$, we can conclude that

$$\begin{aligned} DFT(c[n]) &= DFT(x[n]) + jDFT(y[n]) = F_N^k(x[n]) + jF_N^k(y[n]) \\ \Rightarrow G_N^k(c[n]) &= F_N^k\left(\frac{c[n]+c^*[n]}{2}\right) + jF_N^k\left(\frac{c[n]-c^*[n]}{2}\right) \end{aligned}$$

We could also reach to the above equation by explicitly using the DFT expansion.

ii) By using Problem (6.3) of homework 2, if $C[k]$ is the N-point DFT of the signal $c[n]$, then

$$c[n] \xrightarrow{DFT} Nc[(-k)modN]$$

or equivalently,

$$C[(-n)modN] \xrightarrow{DFT} NC[k].$$

Therefore,

$$c[k] = \frac{1}{N} G_N^k(C[(-n)modN]).$$

iii) Since $G_N^k(\cdot)$ functionals can operate on signals with length at most N, we should divide $x[n]$ into two signals with length N. One way is to divide $x[n]$ to odd and even components :

$$x_e[\ell] = x[2\ell] \text{ and } x_o[\ell] = x[2\ell+1], \text{ for } \ell = 0, 1, \ldots, N-1.$$

Then, for $K = 0, 1, \ldots, 2N-1$,

$$\begin{aligned} X[K] &= \sum_{n=0}^{2N-1} x[n]e^{-j\frac{2\pi}{2N}Kn} = \sum_{\ell=0}^{N-1} x[2\ell]e^{-j\frac{2\pi K}{2N}(2\ell)} + \sum_{\ell=0}^{N-1} x[2\ell+1]e^{-j\frac{2\pi K}{2N}(2\ell+1)} \\ &= \sum_{\ell=0}^{N-1} x_e[\ell]e^{-j\frac{2\pi K}{N}\ell} + e^{-j\frac{\pi K}{N}} \sum_{\ell=0}^{N-1} x_o[\ell]e^{-j\frac{2\pi K}{N}\ell} = X_e[K] + e^{-j\frac{\pi K}{N}} X_o[K]. \end{aligned}$$

Note that the DFT of $x_e[n]$ and $x_o[n]$ is N-point DFT and therefore $X_e[K+N] = X_e[K]$ for $K = 0, 1, \ldots, N-1$. Hence,

$$X[K] = \begin{cases} X_e[K] + e^{-j\frac{\pi K}{N}} X_o[K] & 0 \le K < N \\ X_e[K-N] + e^{-j\frac{\pi K}{N}} X_o[K-N] & N \le K < 2N \end{cases}$$

or

$$X[K] = \begin{cases} G_N^K[x_e[n]] + e^{-j\frac{\pi K}{N}} G_N^K[x_o[n]] & 0 \le K < N \\ G_N^{K-N}[x_e[n]] + e^{-j\frac{\pi K}{N}} G_N^{K-N}[x_o[n]] & N \le K < 2N \end{cases}$$

Problem 3

i) We show that it is not summable. On the contrary, assume that it is summable. Then for every $\varepsilon > 0$, there exists a finite set $J_\varepsilon \in \mathbb{N}$ such that for every finite set $K \in \mathbb{N}$, $\left|\sum_{n\in K} a_n\right| < \varepsilon$ if $K \cap J_\varepsilon = \emptyset$.

We show that it is not possible to have such J_ε. As J_ε is finite set in $\mathbb{N}$, then it has a maximum member. Let $M = \max J_\varepsilon$ then $J_\varepsilon \subseteq \{1, 2, \dots, M\}$. Define $K^{(L)} = \{2M, 2(M+1), \dots, 2(M+L)\}$. $K^{(L)} \cap J_\varepsilon = \emptyset$ for all L values but :

$$\sum_{n\in K^{(L)}} a_n = \frac{1}{2}\sum_{n=M}^{L+M} \frac{1}{n} > \frac{1}{2}\int_M^{M+L} \frac{1}{x}dx + \frac{1}{2M} = \frac{1}{2}\ln\frac{M+L}{M} + \frac{1}{2M}.$$

The inequality could be easily verified similar to what we did in Problem (2.b) in homework 1. Hence,

$$\sum_{n\in K^{(L)}} a_n > \frac{1}{2}\ln(1+\frac{L}{M}) + \frac{1}{2M}.$$

For arbitrary large L, $\left|\sum_{n\in K^{(L)}} a_n\right|$ could take any large value. Therefore, there exists many $K \in \mathbb{N}$ such that $K \cap J_\varepsilon = \emptyset$ and $\sum_{n\in K} a_n > \varepsilon$ for every finite J_ε and $\varepsilon > 0$.

ii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is absolutely convergent if $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent.

$\sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic series and in the Problem (2.b) of homework 1 we have shown that it is divergent. Hence, $\sum \frac{(-1)^n}{n}$ is not absolutely convergent. However, it is convergent.

iii) Let's begin with the sufficient condition. If $\sum_{n=1}^{\infty} a_n$ converge absoutely, then $\{a_n\}$ is summable.

Consider $S_n = \sum_{i=1}^{n} |a_i|$. Since $S_n < \infty$ (it is bounded) and S_n is an increasing sequence, then it converges to a value, call it S. In other words, for every $\varepsilon > 0$,

$$\exists M(\epsilon) \in \mathbb{N} \text{ such that } n \geq M(\epsilon) : \left|\sum_{i=1}^{n} |a_i| - S\right| < \varepsilon$$

For a given $\varepsilon > 0$, let $n = M(\epsilon)$. Thus,

$$\left|\sum_{i=1}^{M(\epsilon)} |a_i| - S\right| = \left|\sum_{i=1}^{M(\epsilon)} |a_i| - \sum_{i=1}^{\infty} |a_i|\right| < \varepsilon$$

But $\sum_{i=1}^{M(\epsilon)} |a_i| - \sum_{i=1}^{\infty} |a_i| = \sum_{i=M(\epsilon)+1}^{\infty} |a_i|$. Therefore, $\sum_{i=M(\epsilon)+1}^{\infty} |a_i| < \varepsilon$. Let $J_\varepsilon = \{1, 2, \dots, M(\epsilon)\}$, then every finite set $K \in \mathbb{N}$ such that $K \cap J_\varepsilon = \emptyset$ is a subset of $\{M(\epsilon)+1, M(\epsilon)+2, \dots\}$ and consequently,

$$\sum_{i\in K} |a_i| < \sum_{i=M(\epsilon)+1}^{\infty} |a_i| < \varepsilon.$$

It means that $\{a_i\}$ is summable.

Now, we prove the necessary condition. If $\{a_n\}$ is summable, then $\sum a_n$ converges absolutely.

Let us first assume that $\{a_n\}$ is a real sequence. If $\{a_n\}$ is summable, for a given $\varepsilon > 0$, there exists J_ε such that for every K finite set and $K \cap J_\varepsilon = \emptyset : \left|\sum_{n\in K} a_n\right| < \varepsilon$.

Consider a given set K. We split it into the finite set K^+ with positive elements and the finite set K^- with negative elements. Therefore, $K^+ \cap J_\varepsilon = \emptyset$ and $K^- \cap J_\varepsilon = \emptyset$. Then :

$$\left.\begin{aligned} \left|\sum_{n\in K^+} a_n\right| &= \sum_{n\in K^+} |a_n| < \varepsilon \\ \left|\sum_{n\in K^-} a_n\right| &= \sum_{n\in K^-} |a_n| < \varepsilon \end{aligned}\right\} \Rightarrow \sum_{n\in K^+\cup K^- = K} |a_n| < 2\varepsilon.$$

Therefore, for every finite set K such that $K \cap J_\varepsilon = \emptyset$:

$$\sum_{n\in K} |a_n| < 2\varepsilon.$$

Since we can take K arbitrarily large, thus $\sum_{n\in\mathbb{N}} |a_n| = \sum_{n\in J_\varepsilon} |a_n| + \sum_{n\notin J_\varepsilon} |a_n|$ is bounded and then it is absolutely convergent.

For the complex sequence, every complex sequence can be written as a summation of two (real and imaginary) sequences :

$$a_n = x_n + jy_n, x_n, y_n \in \mathbb{R}$$

where $|x_n| < |a_n|$ and $|y_n| < |a_n|$ and $|a_n| < |x_n| + |y_n|$.
It can be easily verified that if $\{a_n\}$ is summable then $\{x_n\}$ and $\{y_n\}$ are summable and consequently $\sum x_n$ and $\sum y_n$ are absolutely convergent. Hence, $\sum |a_n| < \sum |x_n| + \sum |y_n| < \infty$ is absolutely convergent.

Problem 4

i) We are looking for the coefficients $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ such that :

$$\alpha_0 + \alpha_1 x + \cdots + \alpha_n x_n = 0 \qquad \forall x \in [0,1]$$

If $\alpha_n \neq 0$, then the above polynomial with degree n has at most n different solutions but it should be zero for all $x \in [0,1]$. Therefore, it is not possible unless all coefficients are equal to zero. Thus, $\{1, x, \ldots, x^n\}$ are linearly independent.

ii) To find the orthonormal basis for a given set of vectors, we use Gram-Schmidt procedure. In the space of $C[0,1]$ define $v_0 = 1, v_1 = x^1, \ldots, v_n = x^n$.
Then $u_0(x) = \frac{v_0}{||v_0||} = 1, ||v_0|| = \sqrt{<1,1>} = \sqrt{\int_0^1 1dx} = 1$

$$u_1(x) = \frac{v_1 - <u_0, v_1> u_0}{||v_1 - <u_0, v_1> u_0||} = \frac{x - <1, x>}{||x - <1, x>||}$$

$$<1, x> = \int_0^1 xdx = \frac{1}{2}, ||x - \frac{1}{2}|| = \sqrt{<x - 1/2, x - 1/2>} = \sqrt{\int_0^1 (x - \frac{1}{2})^2 dx} = \frac{1}{\sqrt{12}}$$

Therefore, $u_1(x) = \sqrt{12}(x - 1/2)$.
The other orthonormal elements of the basis can be made by the following recursion :

$$u_\ell(x) = \frac{x^\ell - \sum_{i=0}^{\ell-1} < x^\ell, u_i(x) > u_i(x)}{||x^\ell - \sum_{i=0}^{\ell-1} < x^\ell, u_i(x) > u_i(x)||}.$$

iii) By the projection theorem, $v_p \in B$ is the projection of a vector v in the Hilbert subspace B, if

$$||v - v_p|| = \inf_{w \in B} ||v - w||.$$

To find a polynomial with degree n, $P_n(x)$, which has the minimum total squared error with $p(x)$, i.e. $||p(x) - P_n(x)||^2$, we should look for the projection of $p(x)$ in the Hilbert space of all polynomial functions of degree n with the norm $|| \cdot ||$.
Assume that $P_n(x) = \sum_{i=0}^{n} b_i u_i(x)$, then :

$$b_i =< P_n(x), u_i(x) >=< p(x), u_i(x) >= \int_0^1 p(x) u_i^*(x) dx.$$

iv) We should find the projection of $\sin \frac{\pi}{2} x$ in the space of polynomials with degree 2. The orthonormal basis $\{u_0(x), u_1(x), u_2(x)\}$ is equal to :

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= \sqrt{12}(x - 1/2), \\ u_2(x) &= \frac{x^2 - < x^2, 1 > - < x^2, u_1(x) > u_1(x)}{||x^2 - < x^2, 1 > - < x^2, u_1(x) > u_1(x)||} = 6\sqrt{5}(x^2 - x + 1/6). \end{aligned}$$

Therefore,

$$\begin{aligned} b_0(x) &=< \sin \frac{\pi}{2} x, 1 >= \int_0^1 \sin \pi/2 x dx = \frac{2}{\pi}, \\ b_1(x) &=< \sin \frac{\pi}{2} x, u_1(x) >= \sqrt{12}\frac{4}{\pi^2} - \sqrt{12}\frac{1}{\pi}, \\ b_2(x) &=< \sin \frac{\pi}{2} x, u_2(x) >= 6\sqrt{5}(\frac{4}{\pi^2} - \frac{16}{\pi^3}) + \frac{2\sqrt{5}}{\pi}. \end{aligned}$$

The two last are concluded since :

$$\begin{aligned} \int_0^1 x \sin \frac{\pi}{2} x dx &= \frac{-2x}{\pi} \cos \frac{\pi}{2} x|_0^1 + \frac{2}{\pi} \int_0^1 \cos \frac{\pi}{2} x dx \\ &= \frac{4}{\pi^2} \sin \frac{\pi}{2} x|_0^1 = \frac{4}{\pi^2}, \\ \int_0^1 x^2 \sin \frac{\pi}{2} x dx &= \frac{-2x^2}{\pi} \cos \frac{\pi}{2} x|_0^1 + \frac{4}{\pi} \int_0^1 x \cos \frac{\pi}{2} x dx \\ &= \frac{8}{\pi^2} x \sin \frac{\pi}{2} x|_0^1 - \frac{8}{\pi^2} \int_0^1 \sin \frac{\pi}{2} x dx \\ &= \frac{8}{\pi^2} + \frac{16}{\pi^3} \cos \frac{\pi}{2} x|_0^1 \\ &= \frac{8}{\pi^2} + \frac{16}{\pi^3}. \end{aligned}$$

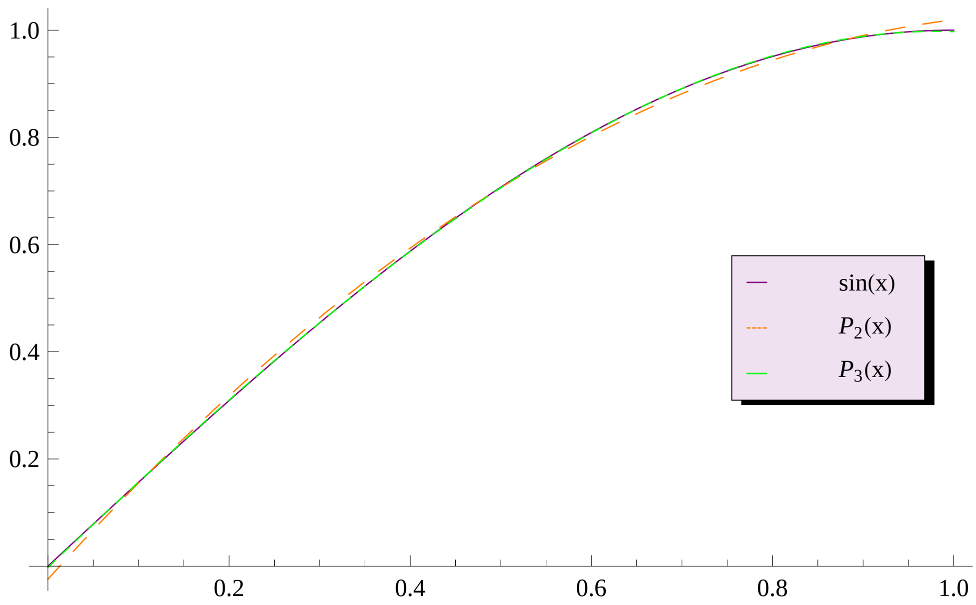

Figure 1: Plot of $\sin \frac{\pi}{2}x$ and its approximated polynomials of degree 2 and 3.

We solved the two integrals by using integration by part. Finally,

$$P_2(x) = b_0 + b_1 u_1(x) + b_2 u_2(x) = -0.024 + 1.878x - 0.834x^2.$$

If we proceed one degree more, the degree 3 approximated polynomial is :

$$P_3(x) = P_2(x) + b_3 u_3(x) = -0.002 + 1.6134x - 0.1724x^2 - 0.4413x^3.$$

In figure 1, the plots of $\sin \frac{\pi}{2}x$, $P_2(x)$ and $P_3(x)$ are depicted. We can see that $P_3(x)$ is located very close to $\sin \frac{\pi}{2}x$.