Signal Processing for Communications

EPFL Spring Semester 2009
April 21, 2009

Midterm

You have 2 hours. We do not necessarily expect that you finish all the problems. Don't lose time on any particular problem, but try to get as many points as you can.

You are allowed to have one piece of A4 paper as crypt sheet. Please, no calculators, cell phones, neighbors, books, notes, formula collections, ...

Good luck!!

Name : ______________________________

Problem I	/ 25
Problem II	/ 25
Problem III	/ 25
Problem IV	/ 25
Total	/100

You might find the following fact useful :

$$x[n] = \begin{cases} 1 & \text{if } |n| \leq T \\ 0 & \text{otherwise} \end{cases} \qquad \overset{\text{DTFT}}{\Longleftrightarrow} \qquad X(e^{j\omega}) = \frac{\sin(\omega(T+\frac{1}{2}))}{\sin(\omega/2)}.$$

<u>Problem I</u> (25 points) Let $x[n]$, $n \in \mathbb{Z}$, be a real valued sequence, and let $X(e^{j\omega})$ denote its DTFT. Suppose that

$$X(e^{j\omega}) = R(4) + R(2)^2 - 5,$$

where we define

$$R(N) = \frac{\sin(\omega(N + \frac{1}{2}))}{\sin(\omega/2)}.$$

$X(e^{j\omega})$ is plotted in Figure 1.

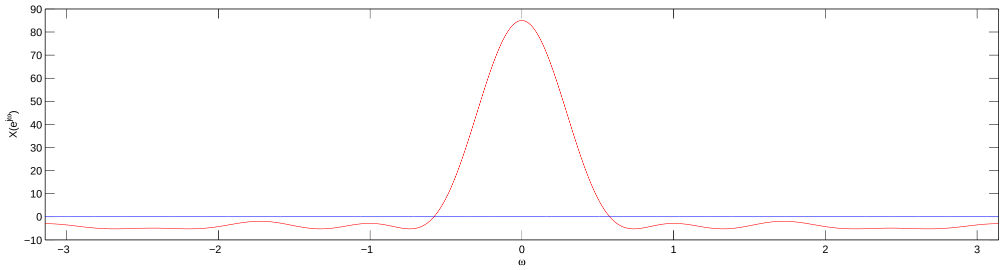

FIG. 1 – $X(e^{j\omega})$

a) [7 pts] Find $x[n]$. Plot its values for $-10 \leq n \leq 10$. HINT : What corresponds to $R(N)^2$?

b) [8 pts] Let $W(e^{j\omega})$ be the DTFT of $x[-n] + x[n]$. Find and plot (approximately) $|W(e^{j\omega})|$.

c) [10 pts] We pass the signal $x[-n+1]$ through an LTI system to obtain $y[n]$ (See Figure 2). Suppose that the impulse response of the system is $h[n] = \delta[n-3]$. Find $Y(e^{j\omega})$, the DTFT of $y[n]$, and plot its phase.

$$x[-n+1] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

FIG. 2 –

<u>Problem II</u> (25 points) Consider and LTI system with the following input-output relation :

$$x[n] + x[n-1] = y[n] - y[n-1] - 4y[n-2] + 4y[n-3]$$

a) [5 pts] Find $H(z)$, the z-transform of the impulse response of this system. Where are the poles and zeros of $H(z)$? HINT : The poles are all integers and very simple.

b) [10 pts] How many different LTI systems with the above relationship between input and output are there? Justify your answer. State which of them are causal and which of them are stable.

c) [10 pts] For the causal one(s), find the impulse response of the system.

Problem III (25 points) Consider the vector $\underline{x} = (x[0], \ldots, x[M-1])$ and let $\underline{X} = (X[0], \ldots, X[M-1])$ denote its DFT. We claim that we can relate $\underline{X}$ to the original signal $\underline{x}$ by a simple matrix multiplication of the form

$$\underline{X} = W\underline{x},$$

for a suitable $M \times M$-matrix W.

a) [5 pts] Specify W.

b) [10 pts] Prove that $W^{-1} = \frac{1}{M}W^H$, where W^H denotes transposition and conjugation (Hermitian).

c) [10 pts] Consider now two vectors $\underline{x}_1$ and $\underline{x}_2$ of length M and let $\underline{X}_1$ and $\underline{X}_2$ denote the result of applying the DFT t times to $\underline{x}_1$ and $\underline{x}_2$, where t is an integer. Prove that

$$\sum_{n=0}^{M-1} x_1^*[n] x_2[n] = \frac{1}{M^t} \sum_{n=0}^{M-1} X_1^*[n] X_2[n].$$

Problem IV **(25 points)** Consider the following discrete-time LTI system with frequency respons $H(e^{j\omega}) = e^{-jd\omega}$, where d is a real number. Let $x[n]$ be the input and $y[n]$ be the output.

a) [8 pts] Let $x[n] = \sin(n\omega_0 + \phi_0)$, where ω_0 and ϕ_0 are real numbers. Let $y[n]$ denote the output of the LTI system. Determine $y[n]$.

b) [7 pts] What is the impulse response of this system if d is an integer ? Is the system BIBO stable ? Is the system causal ?

c) [10 pts] What is the impulse response of this system if d is a real number (but not an integer) ? Is the system BIBO stable ? Is the system causal ?