MIDTERM

Monday November 12, 2007, 13:15-17:00

This exam has 5 problems and 80 points in total.

Instructions

- You are allowed to use 1 sheet of paper for reference. No mobile phones or calculators are allowed in the exam.
- You can attempt the problems in any order as long as it is clear which problem is being attempted and which solution to the problem you want us to grade.
- If you are stuck in any part of a problem do not dwell on it, try to move on and attempt it later.
- Please solve every problem on **separate paper sheets**.
- It is your responsibility to **number the pages** of your solutions and write on the first sheet the **total number of pages** submitted.

GOOD LUCK!

Problem 1

[12 pts]

[*3pts*] (a) Find the DTFT of $x[n] = \frac{1}{2}2^{n+1}u[-n] + j3^{-n}u[n]$.

[*5pts*] (b) Given the same $x[n]$ whose DTFT $X(e^{j\omega})$ we have found in (a), consider the sequence $y[n] = x^*[-n] + x[n]$. Find its DTFT $Y(e^{j\omega})$.

[*4pts*] (c) Given the $x[n]$ whose DTFT $X(e^{j\omega})$ we have found in (a), consider the sequence $z[n] = x^*[1-n] + x[n-1]$. Find its DTFT $Z(e^{j\omega})$.

Problem 2

[18 pts]

Consider the system depicted in Figure 1. The impulse responses of the filters are $h_1[n] = \delta[n-1]$, $h_2[n] = 9\delta[n-1]$, $h_3[n] = 2\delta[n-2]$, $h_4[n] = 3\delta[n-1]$.

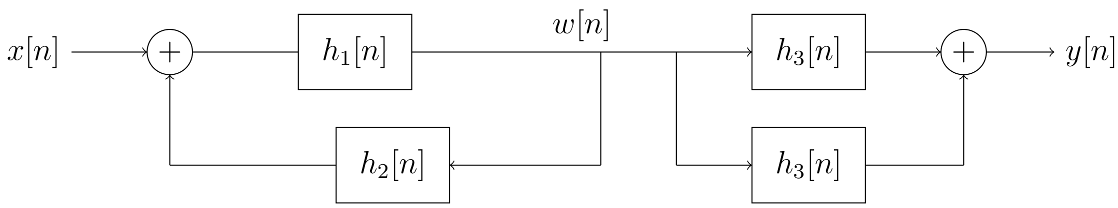

Figure 1: System with Feedback

[*6pts*] (a) Find $H(z)$, the system function of the overall system, and write its corresponding ROC.

[*3pts*] (b) Write down the difference equation corresponding to the overall system.

[*6pts*] (c) Compute the impulse response of the overall system, *i.e.*, find $h[n]$ such that $y[n] = h[n] * x[n]$.

[*1pts*] (d) Is the overall system causal? Why?

[*2pts*] (e) Is the overall system BIBO stable? Prove or disprove.

Problem 3

[12 pts]

Assume that $\tilde{x}[n]$ and $\tilde{y}[n]$ are two periodic sequences with the same period N. Let $\tilde{X}[k]$ and $\tilde{Y}[k]$ be their respective DFS, *i.e.*,

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}, \quad k = 0, \ldots, N-1 \\ \tilde{Y}[k] &= \sum_{n=0}^{N-1} \tilde{y}[n] e^{-j\frac{2\pi}{N}kn}, \quad k = 0, \ldots, N-1.\end{aligned}$$

[*6pts*] (a) Show that

$$\sum_{n=0}^{N-1} \tilde{x}^*[n]\tilde{y}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}^*[k]\tilde{Y}[k].$$

[*6pts*] (b) Suppose,

$$\begin{aligned}\tilde{x}[n] &= 1 + \cos\left(\frac{\pi}{4}n\right) \\ \tilde{y}[n] &= \sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right).\end{aligned}$$

Find the period N for these sequences and compute

$$\sum_{n=0}^{N-1} \tilde{x}^*[n]\tilde{y}[n].$$

Problem 4

[20 pts]

Consider two sequences $x[n]$ and $y[n]$ that have finite-length supports N and M, respectively. More precisely, $x[n] = 0$ for $n < 0$ and $n \geq N$, and $y[n] = 0$ for $n < 0$ and $n \geq M$.
We suppose that $x[n]$ and $y[n]$ are **distinct** sequences, *i.e.*, there exists an n for which $x[n] \neq y[n]$. Assume that $M > N$.

[4pts] (a) Let us take the M-point DFTs, which sample the corresponding DTFTs at M uniformly-spaced points, *i.e.*,

$$\begin{aligned} X_1[k] &= \sum_{n=0}^{M-1} x[n] e^{-j\frac{2\pi}{M}kn}, \quad k = 0, \ldots, M-1 \\ Y_1[k] &= \sum_{n=0}^{M-1} y[n] e^{-j\frac{2\pi}{M}kn}, \quad k = 0, \ldots, M-1. \end{aligned}$$

Suppose we claim that $X_1[k] = Y_1[k]$, $k = 0, \ldots, M-1$. If this can be true, give an example. If not, give a proof.

[8pts] (b) Now, let us sample the corresponding DTFTs at N uniformly spaced points, *i.e.*,

$$\begin{aligned} X_2[k] &= \sum_{n=0}^{M-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad k = 0, \ldots, N-1 \\ Y_2[k] &= \sum_{n=0}^{M-1} y[n] e^{-j\frac{2\pi}{N}kn}, \quad k = 0, \ldots, N-1. \end{aligned}$$

Again, suppose we claim that $X_2[k] = Y_2[k]$, $k = 0, \ldots, N-1$. If this can be true give an example. If not, give a proof.

[8pts] (c) Assume that $M = 2N$, and that

$$y[n] = \begin{cases} x[n], & n = 0, \ldots, N-1 \\ x[n-N], & n = N, \ldots, M-1 \end{cases} \tag{1}$$

and consider the sequence

$$z[n] = y[n] - x[n]. \tag{2}$$

Given that you know that $z[n]$ has been generated as in (1)–(2), we want to find the smallest number of samples of the DTFT of $z[n]$ that would be sufficient to reconstruct $z[n]$. More precisely, suppose the DTFT of $z[n]$, given by

$$Z(e^{j\omega}) = \sum_n z[n] e^{-j\omega n},$$

needs to be sampled as,

$$Z[k] = Z(e^{j\omega})\big|_{\omega = \frac{2\pi k}{Q}}, \quad k = 0, \ldots, Q-1.$$

We want to ensure that $z[n]$ is recoverable from the samples $Z[k], k = 0, \ldots, Q-1$. Find the smallest Q that would be sufficient, given that we know the structure of $z[n]$ as given in (1)–(2). Also, write down the explicit formula of how we can reconstruct $z[n]$, given $Z[k], k = 0, \ldots, Q-1$, for this smallest value of Q.

Problem 5

[18 pts]

Let $H(z) = \frac{8z^{-3}+20z^{-2}-10z^{-1}+3}{8z^{-2}-4z^{-1}+1}$ be the system function of an LTI system.

[*8pts*] (a) Is there a causal system with system function $H(z)$? If yes, give $h[n]$ and say whether the system is stable. If no, show why.

[*3pts*] (b) Is there an anticausal system with system function $H(z)$? If yes, give $h[n]$ and say whether the system is stable. If no, show why.

[*2pts*] (c) Consider the composition of two systems depicted in Figure 2, and assume that when $x[n]$ is the input to the system, then the output is $y[n] = p[n] * x[n] = 8x[n-3] + 20x[n-2] - 10x[n-1] + 3x[n]$. Is the system causal?

[*3pts*] (d) Find the system function $G(z)$ such that $p[n] = h[n] * g[n]$.

[*2pts*] (e) Find the corresponding $g[n]$. Is the system corresponding to $g[n]$ stable?

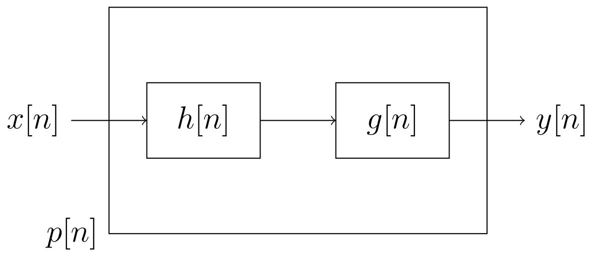

Figure 2: Composition of two Systems