Problem 1 (DFT Revisit). (i) Let \( x[n] \) be a real \( N \)-point sequence and \( X[k] \) be its \( N \)-point DFT. Let \( x_1[n] \) be a sequence such that \( X_1[k] = \text{real}\{X[k](−1)^k\} \). Given that \( N \) is even, find \( x_1[n] \) in terms of \( x[n] \).

(ii) Let \( x_1[n] \) and \( x_2[n] \) be two real \( N \)-point sequences such that \( x_1[n] \) is symmetric, and \( x_2[n] \) is anti-symmetric. Let \( X_1[k] \) and \( X_2[k] \) denote their corresponding \( N \)-point DFTs. Given \( y[n] = x_1[n] + x_2[n] \) with its DFT denoted as \( Y[k] \), explain how \( X_1[k] \) and \( X_2[k] \) can be recovered from \( Y[k] \).

Problem 2 (Limits of Z-transform). Let \( X(z) \) be z-transform of a causal discrete signal \( x[n] \), compute the following equations in terms of \( x[n] \):

(i) \( X(1) \). What does it show?

(ii) \( \lim_{z \to \infty} X(z) \).

(iii) \( \lim_{z \to \infty} z(X(z) − x[0]) \).

(iv) \( −z \frac{dX(z)}{dz} \).

(v) \( \lim_{z \to \infty} −z^2 \frac{dX(z)}{dz} \).

Problem 3 (Stochastic Processes). Consider a discrete random process \( X[n] = \sin(\omega n + \theta) \) such that \( \theta \) is a random variable with uniform distribution on \([0, 2\pi]\) and \( \omega \in \mathbb{R} \).

(i) Find the mean and autocorrelation function of \( X[n] \). Is it a wide-sense stationary process?

Define

\( Y[n] = X[n] + \beta X[n−1] \),

where \( \beta \in \mathbb{R} \).

(ii) Compute the power spectral density \( P_Y(e^{j2\pi f}) \).

Now assume that \( X[n] \) is a zero-mean wide-sense stationary process with autocorrelation function given by

\( R_X[k] = \sigma^2 |\alpha|^k \),

for \( |\alpha| < 1 \).

(iii) Compute the power spectral density \( P_Y(e^{j2\pi f}) \).

(iv) For which values of \( \beta \) does \( Y[n] \) corresponds to a white noise?
Problem 4 (Min. Mean Squared Error Estimator*). In this problem, we want to approximate random variable $X$ in terms of a given set of observations $\{Y_1, \cdots, Y_l\}$ such that $Y_i$, for $1 \leq i \leq l$, is a random variable correlated to $X$.

Consider the case that we have only one observation, $Y$. Define $\hat{X} = h(Y)$ as an estimator of $X$. Then, we estimate the value of $X$ by knowing the value of observation $Y$. An estimator is called **unbiased** estimator, if $\mathbb{E}(X) = \mathbb{E}(\hat{X})$. Assume that $X$ and $Y$ are continuous random variables with the joint probability density function (PDF) $P_{X,Y}(x,y)$. The marginal PDF of $X$ and $Y$ are denoted by $P_X(x)$ and $P_Y(y)$.

Let $\mathcal{H}$ be a Hilbert space of random variables with an inner product defined by

$$
\langle U, V \rangle = \mathbb{E}(UV^*) = \int uv^* P_{X,Y}(x,y) dx dy,
$$

where $U, V \in \mathcal{H}$, the space $\mathcal{H}$ contains $X$ and $Y$ and all the random variables $f(X,Y)$ such that $f(\cdot)$ is a continuous function. Moreover, the subspace of random variable $Y$ contains random variables $Y$ and $f(Y)$ for all continuous function $f(\cdot)$.

i) Prove that $\langle U, V \rangle$ in (1) is an inner product?

Define mean squared error as

$$
\langle x - \hat{x}, x - \hat{x} \rangle = \mathbb{E}(|x - \hat{x}|^2) = \int |x - \hat{x}|^2 P_{X,Y}(x,y) dx dy.
$$

ii) Among the linear unbiased estimators, i.e. $\hat{X} = aY + b$, find the estimator with the minimum mean squared error.

iii) Prove that $\hat{X} = h(Y) = \mathbb{E}(X \mid Y)$ has the minimum mean squared error among all unbiased estimators.

Hint 1: Use projection theorem.

Hint 2: Due to definition of conditional expectation, if $Z = \mathbb{E}(X \mid Y)$ then $\mathbb{E}((X - Z)f(Y)) = 0$ for all continuous function of $f(\cdot)$.

(*) Just for fun. Such problems are out of focus of this course :-).