ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 7 Signal Processing for Communications

Homework 7 April 4, 2011, INF 213 - 10:15-12:00

WRITE THE ANSWER OF EACH PROBLEM ON SEPARATE SHEETS!

This is the second graded homework. You can discuss this homework with your colleagues but you have to write down your own solution. Please indicate all people with whom you have discussed the homework on the top of the first page. If we find similarities beyond the indicated contacts you will get zero points. Please hand in the homework on April 11-th during the exercise session from 10:15 till 12pm. No exceptions.

Problem 1 (The world of Ideals)**.** Consider a system where the input $x[n]$ is first downsampled by a factor $M = 4$, then upsampled by a factor $L = 4$, and finally filtered by a filter $h[n]$. Let the DTFT of the input signal be given as,

$$X(e^{j2\pi f}) = \begin{cases} 8|f| - 1, & \frac{1}{8} \leq |f| \leq \frac{1}{4} \\ 0, & \text{otherwise} \end{cases}$$

Compute and sketch the DTFT of the ouput when $H(e^{jw})$ is given by :

$$H_{\text{LP}}(e^{j2\pi f}) = \begin{cases} 4, & 0 \leq |f| \leq \frac{1}{8} \\ 0, & \text{otherwise} \end{cases}$$

$$H_{\text{BP}}(e^{j2\pi f}) = \begin{cases} 4, & \frac{1}{4} \leq |f| \leq \frac{3}{8} \\ 0, & \text{otherwise} \end{cases}$$

$$H_{\text{HP}}(e^{j2\pi f}) = \begin{cases} 4, & \frac{3}{8} \leq |f| \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Hint: Problem 2 in HW6 migth be useful for you to relate the DTFTs of the downsampled and upsampled signals to the original signal $x[n]$. Recall that downsampling by factor M is defined as $x_d[n] = x[Mn]$, and upsampling by factor L as

$$x_u[n] = \begin{cases} x_d[n/L] & \text{if } n = kL \quad \text{for } k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Problem 2 (Fractional Delay)**.** Consider the following discrete-time LTI system with frequency response $H(e^{j2\pi f}) = e^{-j2\pi f d}$, where d is a real number. Let $x[n]$ be the input and $y[n]$ be the output.

(i) Let $x[n] = sin(2\pi f_0 n + \phi_0)$, where f_0 and ϕ_0 are real numbers. Let $y[n]$ denote the output of the LTI system. Determine $y[n]$.

(ii) What is the impulse response of this system if d is an integer? Is the system BIBO stable? Is the system causal?

(iii) What is the impulse response of this system if d is a real number (but not an integer)? Is the system BIBO stable? Is the system causal?

Problem 3 (Be Rational!). The transfer function of an LTI system is given by

$$H(z) = \frac{(1 - \frac{1}{3}z^{-1})}{(1 + \frac{1}{2}z^{-1})(1 - z^{-1} + \frac{1}{2}z^{-2})}$$

1. Find the difference equation which characterizes this system.
2. Plot the pole-zero diagram. Describe all valid ROCs that can be associated with $H(z)$. For each region, state whether the system is causal and stable.
3. Using only the diagram, plot approximately the magnitude response of $H(e^{j2\pi f})$ inside the regions where the Fourier transform converges.
4. Find the inverse z-transform corresponding to each ROC you found in part 2).
   Hint: The following z-transfrom pair migth be useful,
   $[r^n \cos(w_0 n)u[n]] \xrightarrow{Z} \frac{1 - [r\cos(w_0)]z^{-1}}{1 - [2r\cos(w_0)]z^{-1} + r^2 z^{-2}}$, for $|z| > |r|$.
5. Suppose we would like to find a transfer function $G(z)$ which cancels the effect of $H(z)$, i.e. $H(z)G(z) = 1$.
   (i) Plot the pole-zero diagram of $G(z)$.
   (ii) Can you identify a region inside which both $H(z)$ and $G(z)$ are the transfer functions of stable and causal systems? If so, give the ROC of the overall system, and plot approximately the magnitude response of $G(e^{j2\pi f})$.

Problem 4 (FIR Approximation of the Hilbert Filter).

1. A system is characterized as a generalized linear phase if its frequency response has a constant group delay, i.e, $\arg\{H(e^{j2\pi f})\} = \phi - 2\pi f d$, where $0 < f < \frac{1}{2}$ and ϕ, $d \in \mathbb{R}$, and $\frac{d}{df}\arg\{H(e^{j2\pi f})\} = 2\pi d$ constant. Note that if the impulse response of a causal FIR filter is symmetric/antisymmetric then it has generalized linear phase.
   (i) Assume M is odd, show that the $M+1$ length impulse response $H_s(e^{j2\pi f})$ of a symmetric, causal FIR filter $h_s[n]$ is of the form:

   $$H_s(e^{j2\pi f}) = e^{-j2\pi f \frac{M}{2}} \sum_{k=1}^{\frac{M+1}{2}} b[k]\cos(2\pi f(k - \frac{1}{2}))$$

   where $b[k] = 2h_s[\frac{M+1}{2} - k]$, for $k = 1, 2, \ldots, \frac{M+1}{2}$.
   (ii) Assume M is odd, show that the $M+1$ length impulse response $H_{as}(e^{jw})$ of an antisymmetric, causal FIR filter $h_{as}[n]$ is of the form:

   $$H_{as}(e^{j2\pi f}) = je^{-j2\pi f \frac{M}{2}} \sum_{k=1}^{\frac{M+1}{2}} d[k]\sin(2\pi f(k - \frac{1}{2}))$$

   where $d[k] = 2h_{as}[\frac{M+1}{2} - k]$, for $k = 1, 2, \ldots, \frac{M+1}{2}$.

2. We would like to obtain a causal FIR filter approximation of the Hilbert filter such that the obtained filter has generalized linear phase. We denote the approximated impulse response by $h[n]$.

   (i) Assume we first introduce a constant group delay and obtain the filter $h_d[n]$. Give the expression of the frequency response $H_d(e^{j2\pi f})$. Plot its magnitude and phase.

   (ii) Can you use $h_s[n]$ and $h_{as}[n]$ to approximate $h_d[n]$? Explain.

   (iii) Assume we would like to use windowing operation for the approximation. Let $w[n]$ be a window of length $M+1$ (M is odd), i.e. $h[n] = h_d[n]w[n]$. Determine $h_d[n]$ in this case.

   (iv) Find the optimal window of length M which minimizes the mean square error of the frequency response, i.e.,

$$\epsilon^2 = \int_{-1/2}^{1/2} |H(e^{j2\pi f}) - H_d(e^{j2\pi f})|^2 \mathrm{d}f$$

Note: Check Discrete-Time Signal Processing book by Oppenheim et al. for more details.

Problem 5 (Just for Fun!)**.** Consider the following signals x[n] and y[n] :

$$x[n] = \cos(\frac{\pi}{49}n + \frac{\pi}{17}),$$
$$y[n] = \begin{cases} \frac{|49-n|}{49}, & -49 \leq n \leq 49 \\ 0, & \text{otherwise} \end{cases}$$

What is $z[n] = x[n] * y[n]$?