ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 9 Signal Processing for Communications
Homework 5 March 21, 2011, INF 213 - 10:15-12:00

Problem 1 (Golay Sequences)**.**

Consider the sequence $\mathbf{x} = (x_0, \cdots, x_N)$ where $x_i \in \{-1, 1\}$. The aperiodic autocorrelation function of the sequence $\mathbf{x}$ is defined as follows,

$$R_x(k) = \sum_{i=0}^{N-k-1} x_i x_{i+k}, \quad \text{for } k = 0, \cdots, N-1.$$

It is easy to check that $R_x(0) = N$. We say that the pair of sequences $\mathbf{x}, \mathbf{y} \in \{-1, +1\}^N$ are complementary, which we denote by $\mathbf{x} \sim \mathbf{y}$, if

$$R_x(k) + R_y(k) = 2N\delta[k],$$

where $\delta[k] = 1$ for $k = 0$ and zero otherwise.

i) Let $\mathbf{x} \sim \mathbf{y}$. Show that

   a) $\mathbf{a} \sim \mathbf{b}$, where $a_k = (-1)^k x_k$ and $b_k = (-1)^k y_k$.

   b) $\hat{\mathbf{x}} \sim \mathbf{y}$, where $\hat{x}_k = x_{N-k-1}$.

ii) Let $x[n]$ be a discrete signal such that $x[k] = x_k$ for $k = 0, \cdots, N-1$ and zero otherwise. By using Parseval theorem prove that $\max_f \left|X(e^{j2\pi f})\right| \geq \sqrt{N}$ where $X(e^{j2\pi f})$ denotes the discrete time Fourier transform of $x[n]$.

iii) If $\mathbf{x} \sim \mathbf{y}$, prove that

$$|X(e^{j2\pi f})|, |Y(e^{j2\pi f})| \leq \sqrt{2N}.$$

Hint: First show that $\left|X(e^{j2\pi f})\right|^2 + \left|Y(e^{j2\pi f})\right|^2 = 2N$.

Problem 2 (LTI Systems)**.**

For each of the following systems determine whether the system is (1) linear, (2) time-invariant, (3) stable, (4) causal, and (5) memoryless.

i) $T\{x[n]\} = \displaystyle\sum_{k=n_0}^{n} x[k]$.

ii) $T\{x[n]\} = x[Mn]$ where M is a positive integer.

iii) $T\{x[n]\} = x[n] * x[n]$.

iv) $T\{x[n]\} = \mathit{median}\{x[n-M_1], \ldots, x[n-1], x[n], x[n+1], \ldots, x[n+M_2]\}$ where M_1 and M_2 are positive integers.

v) $T\{x[n]\} = x[n] * h_1[n] * h_2[n] * h_3[n]$ where

$$x[n] * h_1[n] = \begin{cases} x[n/M] & \text{when n is a multiple of } M \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] * h_2[n] = x[n] - \tfrac{1}{2}x[n-1]$$

$$x[n] * h_3[n] = x[Mn]$$

and M is a positive integer.

Problem 3 (Z-Transform)**.**

Let $X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$ be defined as the z-transform of the sequence x[n]. Find the z-transform of the following two sequences, and draw their region of convergences (ROCs).

i) $x[n] = a^n u[n]$

ii) $x[n] = -a^n u[-n-1]$

Which of the above two sytems are causal?

Problem 4 (Inverse Z-Transform)**.**

Power Series Expansion: If we expand the z-transform summation, we obtain:

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n} = \cdots + x[-1]z^1 + x[0]z^0 + x[1]z^{-1} + \ldots$$

We observe that the coefficients of this expansion are the sequence values of $x[n]$. Hence in case the ROC is appropriately specified, you might be able to invert a given $X(z)$ uniquely by finding its power series expansion.

1) Find the inverse z-transform using power series expansion.

i) $X(z) = z + \dfrac{(1-z^{-2})(1+\frac{1}{2}z^{-1})}{(1+z)}$, $0 < |z| < \infty$.

ii) $X(z) = \log(1-2z^{-1})$, $|z| > 2$.

Partial Fraction Expansion: Assume that the z-transform can be represented as a ratio of two polynomials, i.e.

$$X(z) = \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

where c_k's are the zeros and d_k's the poles of $X(z)$. When $M < N$ and all the d_k's are distinct, the partial fraction expansion reduces $X(z)$ into the following form:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

where $A_k = (1 - d_k z^{-1})X(z)|_{z=d_k}$. As you saw in Problem 3, $X(z)$ alone does not uniquely specify the corresponding $x[n]$. If the given ROC of $X(z)$ is such that each fraction of the above summation corresponds to the z-transform of a causal or anti-causal sequence, then you will be able to invert $X(z)$ into a unique sequence $x[n]$.

2) Find the inverse z-transform using partial fraction expansion.

i) $X(z) = \dfrac{1}{(1 - \frac{1}{7}z^{-1})(1 - 5z^{-1})}, \qquad \dfrac{1}{7} < |z| < 5$

ii) $X(z) = \dfrac{1 + \frac{1}{2}z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}, \qquad |z| > \frac{1}{2}$

Contour integration: The inverse z-transform can be computed directly by evaluating the following integral:

$$x[n] = \frac{1}{i2\pi} \oint_C X(z) z^{n-1} \mathrm{d}z$$

where the integration is around a counterclockwise closed circular contour of radius $|z| = r$ inside the ROC.

3) Find the inverse z-transform evaluating the contour integral.

i) $X(z) = \dfrac{(1 - \frac{1}{2}z^{-1})}{(1 - \frac{1}{4}z^{-1})(1 + z^{-1})}, \ |z| > 1$

Hint. Use the residue integration method.