Problem 1 (Golay Sequences).

Consider the sequence \( x = (x_0, \ldots, x_N) \) where \( x_i \in \{-1, 1\} \). The aperiodic autocorrelation function of the sequence \( x \) is defined as follows,

\[
R_x(k) = \sum_{i=0}^{N-k-1} x_i x_{i+k}, \quad \text{for } k = 0, \ldots, N - 1.
\]

It is easy to check that \( R_x(0) = N \). We say that the pair of sequences \( x, y \in \{-1, +1\}^N \) are complementary, which we denote by \( x \sim y \), if

\[
R_x(k) + R_y(k) = 2N\delta[k],
\]

where \( \delta[k] = 1 \) for \( k = 0 \) and zero otherwise.

i) Let \( x \sim y \). Show that

a) \( a \sim b \), where \( a_k = (-1)^k x_k \) and \( b_k = (-1)^k y_k \).

b) \( \hat{x} \sim y \), where \( \hat{x}_k = x_{N-k-1} \).

ii) Let \( x[n] \) be a discrete signal such that \( x[k] = x_k \) for \( k = 0, \ldots, N - 1 \) and zero otherwise. By using Parseval theorem prove that \( \max_f |X(e^{j2\pi f})| \geq \sqrt{N} \) where \( X(e^{j2\pi f}) \) denotes the discrete time Fourier transform of \( x[n] \).

iii) If \( x \sim y \), prove that

\[
|X(e^{j2\pi f})|, |Y(e^{j2\pi f})| \leq \sqrt{2N}.
\]

Hint: First show that \( |X(e^{j2\pi f})|^2 + |Y(e^{j2\pi f})|^2 = 2N \).

Problem 2 (LTI Systems).

For each of the following systems determine whether the system is (1) linear, (2) time-invariant, (3) stable, (4) causal, and (5) memoryless.

i) \( T\{x[n]\} = \sum_{k=n_0}^{n} x[k] \).

ii) \( T\{x[n]\} = x[Mn] \) where \( M \) is a positive integer.

iii) \( T\{x[n]\} = x[n] * x[n] \).

iv) \( T\{x[n]\} = \text{median}\{x[n-M_1], \ldots, x[n-1], x[n], x[n+1], \ldots, x[n+M_2]\} \) where \( M_1 \) and \( M_2 \) are positive integers.
v) \( T\{x[n]\} = x[n] * h_1[n] * h_2[n] * h_3[n] \) where

\[
x[n] * h_1[n] = \begin{cases} 
  x[n/M] & \text{when } n \text{ is a multiple of } M \\
  0 & \text{otherwise}
\end{cases}
\]

\[
x[n] * h_2[n] = x[n] - \frac{1}{2}x[n - 1]
\]

\[
x[n] * h_3[n] = x[Mn]
\]

and \( M \) is a positive integer.

**Problem 3 (Z-Transform).**

Let \( X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \) be defined as the \( z \)-transform of the sequence \( x[n] \). Find the \( z \)-transform of the following two sequences, and draw their region of convergences (ROCs).

i) \( x[n] = a^n u[n] \)

ii) \( x[n] = -a^n u[-n - 1] \)

Which of the above two systems are causal?

**Problem 4 (Inverse Z-Transform).**

**Power Series Expansion:** If we expand the \( z \)-transform summation, we obtain:

\[
\sum_{n=-\infty}^{\infty} x[n]z^{-n} = \cdots + x[-1]z^1 + x[0]z^0 + x[1]z^{-1} + \ldots
\]

We observe that the coefficients of this expansion are the sequence values of \( x[n] \). Hence in case the ROC is appropriately specified, you might be able to invert a given \( X(z) \) uniquely by finding its power series expansion.

1) Find the inverse \( z \)-transform using power series expansion.

i) \( X(z) = z + \frac{(1 - z^{-2})(1 + \frac{1}{2}z^{-1})}{(1 + z)}, \) \( 0 < |z| < \infty. \)

ii) \( X(z) = \log(1 - 2z^{-1}), |z| > 2. \)

**Partial Fraction Expansion:** Assume that the \( z \)-transform can be represented as a ratio of two polynomials, i.e.

\[
X(z) = \prod_{k=1}^{M} \frac{1 - c_k z^{-1}}{\prod_{k=1}^{N} (1 - d_k z^{-1})}
\]
where $c_k$’s are the zeros and $d_k$’s the poles of $X(z)$. When $M < N$ and all the $d_k$’s are distinct, the partial fraction expansion reduces $X(z)$ into the following form:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

where $A_k = (1 - d_k z^{-1})X(z)|_{z=d_k}$. As you saw in Problem 3, $X(z)$ alone does not uniquely specify the corresponding $x[n]$. If the given ROC of $X(z)$ is such that each fraction of the above summation corresponds to the $z$-transform of a causal or anti-causal sequence, then you will be able to invert $X(z)$ into a unique sequence $x[n]$.

2) Find the inverse $z$-transform using partial fraction expansion.

i) $X(z) = \frac{1}{(1 - \frac{1}{7} z^{-1})(1 - 5 z^{-1})}, \quad \frac{1}{7} < |z| < 5$

ii) $X(z) = \frac{1 + \frac{1}{2} z^{-2}}{1 - \frac{3}{4} z^{-1} + \frac{13}{8} z^{-2}}, \quad |z| > \frac{1}{2}$

**Contour integration:** The inverse $z$-transform can be computed directly by evaluating the following integral:

$$x[n] = \frac{1}{i2\pi} \oint_{C} X(z) z^{n-1} \, dz$$

where the integration is around a counterclockwise closed circular contour of radius $|z| = r$ inside the ROC.

3) Find the inverse $z$-transform evaluating the contour integral.

i) $X(z) = \frac{1 - \frac{1}{7} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 + z^{-1})}, \quad |z| > 1$

*Hint. Use the residue integration method.*