ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 1	Signal Processing for Communications
Homework 1	February 21, 2011, BC04 - 10:15-12:00

Problem 1 (Continuity of a function)**.** Recall that a function $f(x)$ is continuous at x_0 if for every $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \quad \text{whenever } |x - x_0| < \delta(\epsilon).$$

It means that $\lim_{x \to x_0} f(x) = f(x_0)$.

a) Let $sinc(x) = \frac{sin(\pi x)}{(\pi x)}$ for $x \neq 0$, $sinc(0) = 1$. Show that $sinc(x)$ is continuous everywhere.
Hint: $r(x) = \frac{p(x)}{q(x)}$ is continuous at x_0, if $p(x)$ and $q(x)$ are continuous at x_0 and $q(x_0) \neq 0$.

b) Define $g(x)$ as follows:

$$g(x) = \begin{cases} 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 \text{ if } x \in \mathbb{Q} \end{cases}$$

where $\mathbb{R}$ denotes the set of real numbers and $\mathbb{Q}$ denotes the set of rational numbers. Show that $g(x)$ is nowhere continuous.

c) Assume that $f(x)$ is continuous at x_0 and $g(y)$ is continuous at y_0, where $y_0 = f(x_0)$. Then prove that the composite function $h = g \circ f$ is continuous at x_0.

Problem 2 (Convergence of infinite series)**.** Let $S_n = \sum_{i=1}^{n} a_i$ where $a_i \in \mathbb{R}$. We say that the series $\sum_{i=1}^{\infty} a_i$ is *convergent* and has the sum S, if $\lim_{n\to\infty} S_n = S$, i.e., for every $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that

$$|S_n - S| < \epsilon \quad \text{for all } n > N.$$

a) (*Necessary condition*) Show that if $\sum_{i=1}^{\infty} a_i$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

b) Let $\sum_{i=1}^{\infty} \frac{1}{i^p}$. Verify for which values of $p \in \mathbb{R}^+$, the series is convergent.
Hint: Let $s(x) = \frac{1}{x^p}$. Then prove that

$$\sum_{i=2}^{n} \frac{1}{i^p} \leq \int_1^n s(x)\mathrm{d}x \leq \sum_{i=1}^{n-1} \frac{1}{i^p}.$$

c) Test the following series for convergence or divergence.

i) $\displaystyle\sum_{n=1}^{\infty} x^n$ (Find a condition on the value of x for convergence).

ii) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

iv) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(*Cauchy criterion for convergence*) The series $\sum_{i=1}^{\infty} a_i$ converges if and only if for every $\epsilon > 0$, there is a number $N \in \mathbb{N}$ such that $|\sum_{i=n}^{m} a_i| < \epsilon$ for all $m \geq n > N$.

d) (*Absolute convergence*) Show that convergence of the series $\sum_{i=1}^{\infty} |a_i|$ implies convergence of $\sum_{i=1}^{\infty} a_i$.

Problem 3 (Sums)**.** Compute the following sums (Check finiteness of the sums if necessary).

i) $\displaystyle\sum_{k=i}^{n} x^k$.

ii) $\displaystyle\sum_{k=1}^{n} kx^k$.

iii) $\displaystyle\sum_{n=1}^{\infty} (\frac{\sqrt{3}}{2} + \frac{1}{j2})^n$.

iv) $\displaystyle\sum_{k=1}^{n} sin(2\pi \frac{k}{N}), \qquad n < N$.

Problem 4 (Inner Product Properties)**.** Let E be an inner product space over $\mathbb{C}$. Let $x, y \in E$ and define $\|x\| =< x, x >^{\frac{1}{2}}$.

a) Show that

$$\|x + y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\,\{< x, y >\} + \|y\|^2$$

holds. Assume that $E = \mathbb{R}^2$, and x, y are orthogonal vectors. Do you recover a familiar formula?

b) (*Parallelogram Law*) Show that

$$2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2$$

holds. Give a geometrical interpretation when $E = \mathbb{R}^2$.

c) (*Polarization Identity*) Show that

$$< x, y >= \frac{1}{4}\{\|x + y\|^2 - \|x - y\|^2 + j\|x + jy\|^2 - j\|x - jy\|^2\}$$

holds. Check that the above definition does satisfy the properties of an inner product. You can assume the scaling property holds, i.e. $< \alpha x, y >= \alpha < x, y >$ for any $\alpha \in \mathbb{C}$.