Problem 1 (Continuity of a function). Recall that a function \( f(x) \) is continuous at \( x_0 \) if for every \( \epsilon > 0 \), there is a \( \delta(\epsilon) > 0 \) such that

\[
|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta(\epsilon).
\]

It means that \( \lim_{x \to x_0} f(x) = f(x_0) \).

a) Let sinc \( x = \frac{\sin(\pi x)}{\pi x} \) for \( x \neq 0 \), sinc \( (0) = 1 \). Show that sinc \( x \) is continuous everywhere.

Hint: \( r(x) = \frac{p(x)}{q(x)} \) is continuous at \( x_0 \), if \( p(x) \) and \( q(x) \) are continuous at \( x_0 \) and \( q(x_0) \neq 0 \).

b) Define \( g(x) \) as follows:

\[
g(x) = \begin{cases} 
0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\
1 & \text{if } x \in \mathbb{Q}
\end{cases}
\]

where \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{Q} \) denotes the set of rational numbers. Show that \( g(x) \) is nowhere continuous.

c) Assume that \( f(x) \) is continuous at \( x_0 \) and \( g(y) \) is continuous at \( y_0 \), where \( y_0 = f(x_0) \). Then prove that the composite function \( h = g \circ f \) is continuous at \( x_0 \).

Problem 2 (Convergence of infinite series). Let \( S_n = \sum_{i=1}^{n} a_i \) where \( a_i \in \mathbb{R} \). We say that the series \( \sum_{i=1}^{\infty} a_i \) is convergent and has the sum \( S \), if \( \lim_{n \to \infty} S_n = S \), i.e., for every \( \epsilon > 0 \), there is a \( N \in \mathbb{N} \) such that

\[
|S_n - S| < \epsilon \quad \text{for all } n > N.
\]

a) (Necessary condition) Show that if \( \sum_{i=1}^{\infty} a_i \) is convergent, then \( \lim_{n \to \infty} a_n = 0 \).

b) Let \( \sum_{i=1}^{\infty} \frac{1}{i^p} \). Verify for which values of \( p \in \mathbb{R}^+ \), the series is convergent.

Hint: Let \( s(x) = \frac{1}{x^p} \). Then prove that

\[
\sum_{i=2}^{n} \frac{1}{i^p} \leq \int_{1}^{n} s(x)dx \leq \sum_{i=1}^{n-1} \frac{1}{i^p}.
\]

c) Test the following series for convergence or divergence.

i) \( \sum_{n=1}^{\infty} x^n \) (Find a condition on the value of \( x \) for convergence).

ii) \( \sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^n \).

iii) \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \).
iv) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \).

(Cauchy criterion for convergence) The series \( \sum_{i=1}^{\infty} a_i \) converges if and only if for every \( \epsilon > 0 \), there is a number \( N \in \mathbb{N} \) such that \( |\sum_{i=n}^{m} a_i| < \epsilon \) for all \( m \geq n > N \).

**d) (Absolute convergence)** Show that convergence of the series \( \sum_{i=1}^{\infty} |a_i| \) implies convergence of \( \sum_{i=1}^{\infty} a_i \).

**Problem 3** (Sums). Compute the following sums (Check finiteness of the sums if necessary).

i) \( \sum_{k=i}^{n} x^k \).

ii) \( \sum_{k=1}^{n} kx^k \).

iii) \( \sum_{n=1}^{\infty} \left( \frac{\sqrt{3}}{2} + \frac{1}{j2} \right)^{2n} \).

iv) \( \sum_{k=1}^{n} \sin(2\pi \frac{k}{N}), \quad n < N \).

**Problem 4** (Inner Product Properties). Let \( E \) be an inner product space over \( \mathbb{C} \). Let \( x, y \in E \) and define \( \|x\| = \langle x, x \rangle^{\frac{1}{2}} \).

a) Show that \( \|x + y\|^2 = \|x\|^2 + 2 \text{Re} \{\langle x, y \rangle\} + \|y\|^2 \) holds. Assume that \( E = \mathbb{R}^2 \), and \( x, y \) are orthogonal vectors. Do you recover a familiar formula?

b) (Parallelogram Law) Show that \( 2\|x\|^2 + 2\|y\|^2 = \|x + y\|^2 + \|x - y\|^2 \) holds. Give a geometrical interpretation when \( E = \mathbb{R}^2 \).

c) (Polarization Identity) Show that \( \langle x, y \rangle = \frac{1}{4} \left\{ \|x + y\|^2 - \|x - y\|^2 + j\|x + jy\|^2 - j\|x - jy\|^2 \right\} \) holds. Check that the above definition does satisfy the properties of an inner product. You can assume the scaling property holds, i.e. \( \langle \alpha x, y \rangle = \alpha \langle x, y \rangle \) for any \( \alpha \in \mathbb{C} \).