

Problem 1 (Gibbs Phenomenon). In this exercise we will demonstrate the Gibbs phenomenon through rectangular windowing. Suppose we want to design a low pass filter with a cut-off frequency of $1/4$, i.e. we have a desired frequency response

$$H(e^{j2\pi f}) = \begin{cases} 1 & -\frac{1}{4} \leq f \leq \frac{1}{4} \\ 0 & \text{Otherwise} \end{cases}$$

Let $h[n]$ be the corresponding impulse response. We want to create a $2N + 1$ taps filter that represents the desired response as close as possible. We do this by applying a rectangular window $w[n]$ that is defined by

$$w[n] = \begin{cases} 1 & -N \leq n \leq N \\ 0 & \text{Otherwise} \end{cases}$$

The resulting filter is given by

$$\hat{h}[n] = h[n]w[n].$$

The goal of this exercise is to see the difference between $|\hat{H}(e^{j2\pi f})|$ and $|H(e^{j2\pi f})|$.

- i) Obtain the impulse response of $h[n]$.
- ii) Write a function *LowPass.m* which take N and return the vector $\hat{h}[n]$.
Hint: Note that $h[n]$ and $w[n]$ are non-causal but your function *LowPass.m* will be causal. Find a proper way to solve this issue. We made *LowPass.m* in problem 2 of HW2.
- iii) Plot $|\hat{H}(e^{j2\pi f})|$ for $N = 10, 100, 500, 1000, 10000$. How does the maximum of $|\hat{H}(e^{j2\pi f})|$, for $-\frac{1}{2} \leq f \leq \frac{1}{2}$, depend on N ?
- iv) By using *fft*, compute $W(e^{j2\pi f})$.
Hint : take the length of the signal $w[n]$ much larger than N .
- v) We know that

$$\hat{H}(e^{j2\pi f}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} H(e^{j2\pi(f-\theta)})W(e^{j2\pi\theta})d\theta.$$

Write a program that for a given N , find the frequency f_N that $|\hat{H}(e^{j2\pi f})|$ gets maximized, then plot $H(e^{j2\pi(f-\theta)})W(e^{j2\pi\theta})$ and $W(e^{j2\pi\theta})$ in terms of $-\frac{1}{2} \leq \theta \leq \frac{1}{2}$ in the same figure. Run the program for $N = 10, 100, 1000, 10000$. What can you say about f_N ?

- vi) Justify why the maximum of $|\hat{H}(e^{j2\pi f})|$ remains fixed by changing N ?

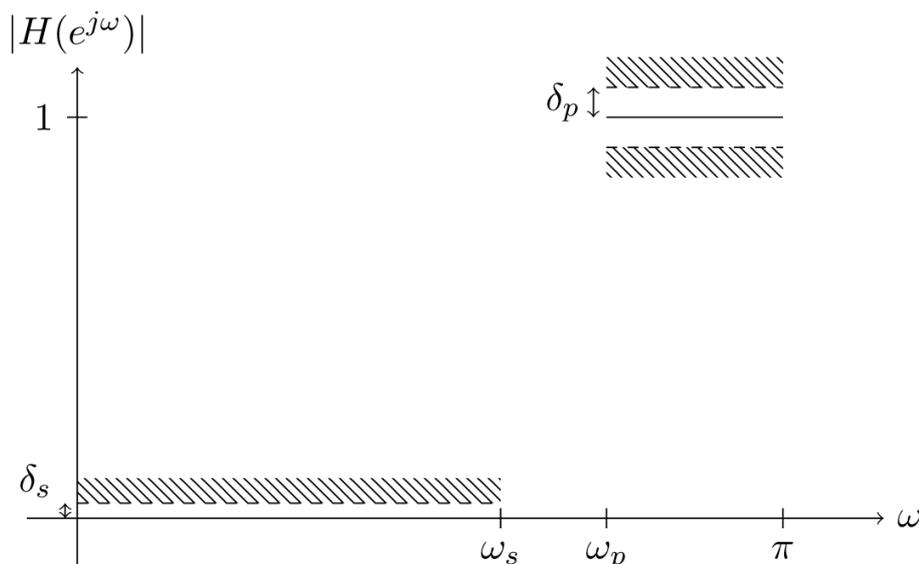


Figure 1: Filter specification for Problem 2.

Problem 2 (Weighted Least-Squares Filter Design). The goal of this problem is to design a filter that minimizes a weighted mean square error with respect to an ideal highpass filter in the passband and the stopband, with no requirements for the transition band. We want the filter to satisfy certain tolerances. In the following you will design a symmetric FIR filter $h[n]$ according to the specifications in Fig. 1. The passband is $[f_p, \frac{1}{2}]$, the stopband is $[0, f_s]$, and the transition band is (f_s, f_p) . The tolerances are, respectively, δ_p for the passband and δ_s for the stopband. Recall that the centered impulse response $h_d[n]$ of a symmetric FIR filter with odd M taps satisfies

$$H_d(e^{j2\pi f}) = h_0 + 2 \sum_{n=1}^{(M-1)/2} h_d[n] \cos(2\pi n f).$$

The error function $E(f)$ is defined as

$$E(f) = W(e^{j2\pi f})[D(e^{j2\pi f}) - H_d(e^{j2\pi f})],$$

where

$$D(e^{j2\pi f}) = \begin{cases} 0 & |f| \in [0, f_s] \\ 1 & |f| \in [f_p, \frac{1}{2}] \end{cases}$$

is the desired (ideal) highpass filter frequency response and

$$W(e^{j2\pi f}) = \begin{cases} \delta_p/\delta_s & |f| \in [0, f_s] \\ 1 & |f| \in [f_p, \frac{1}{2}] \\ 0 & \text{Otherwise} \end{cases}$$

is the weighting function. We want to find the filter coefficients that minimize the squared error, i.e.,

$$\{h_d[n]\}_{n=0}^{(M-1)/2} = \arg \min_{h_d[n]} \left\{ \int_0^{\frac{1}{2}} |E(f)|^2 df \right\},$$

subject to

$$|E(f)| \leq \delta_p \quad \text{for all } f.$$

This filter design method is called *weighted least squares* or WLS.

For a discrete set of frequencies the objective function is approximated by

$$\int_0^{\frac{1}{2}} |E(f)|^2 df \approx \frac{1}{K} \sum_{k=1}^K |E(f_k)|^2, \text{ for } f_k = k/K. \quad (1)$$

- i) Let $\mathbf{e} = [E(f_1), \dots, E(f_K)]^T$. Then $\sum_{k=1}^K |E(f_k)|^2 = \mathbf{e}^T \mathbf{e}$. Find the matrices \mathbf{W} and \mathbf{U} and the vector \mathbf{d} such that we can write

$$\mathbf{e} = \mathbf{W}(\mathbf{d} - \mathbf{U}\mathbf{h}),$$

where $\mathbf{h} = [h_d[0], h_d[1], \dots, h_d[(M-1)/2]]^T$ are the coefficients of the centered filter impulse response.

The filter design problem can now be formulated as

$$\min_{\mathbf{h}} \mathbf{e}^T \mathbf{e},$$

subject to

$$|e_k| \leq \delta_p, \quad k = 1, \dots, K. \quad (2)$$

This kind of problem is called a *nonlinear constrained optimization problem* and is standard in optimization theory.

A general nonlinear constrained optimization problem has the form

$$\begin{aligned} &\text{minimize } f(\mathbf{h}) \\ &\text{s.t. } \mathbf{A}\mathbf{h} \leq \mathbf{b}, \end{aligned} \quad (3)$$

where $f(\cdot)$ is a scalar function of a vector argument and where the inequality must be satisfied for all components of the respective vectors. In our case, we have $f(\mathbf{h}) = \mathbf{e}^T \mathbf{e}$.

- ii) Find the matrix \mathbf{A} and the vector \mathbf{b} such that the constraint (2) can be written as $\mathbf{A}\mathbf{h} \leq \mathbf{b}$. Express \mathbf{A} and \mathbf{b} in terms of \mathbf{W} , \mathbf{U} , and \mathbf{d} .

To solve a problem of the form (3), MATLAB provides the function *fmincon* (which is part of the *optimization toolbox*). In its basic form, its syntax is

$$[\mathbf{x}, fval] = \text{fmincon}(f, \mathbf{x}_0, \mathbf{A}, \mathbf{b})$$

Here f is a handle to the objective function, \mathbf{x}_0 is an initial guess for \mathbf{x} , and \mathbf{A} and \mathbf{b} define the inequality constraint $\mathbf{A}\mathbf{h} \leq \mathbf{b}$. *fmincon* has two return values: \mathbf{x} is the argument that minimizes the objective function; *fval* is the function value at that point.

A handle to a function, or function handle, is a way to pass a reference to some function to another function. For example, suppose that you have written a MATLAB function *myfunc* that takes a vector \mathbf{x} and returns the value of the objective function for \mathbf{x} . Then you would call *fmincon* as

$$\text{fmincon}(@f, \dots).$$

Thus, to create a handle to a function you simply precede the function's name with the @ sign.

- iii) Complete the following code and Use the values for f_p , f_s , etc. defined in the code. Provide a plot of $|H(e^{j2\pi f})|$ as well as a stem plot of $h[n]$.

```
function h = wls()
% WLS   Weighted least square filter design
%   H = WLS() returns the coefficients for a symmetric, causal FIR
%   highpass filter, designed using weighted least squares.

% Disable a particular warning to avoid an annoying warning message
warning off optim:fmincon:SwitchingToMediumScale;

% Filter specifications

f_s = 0.19;          % End of stopband frequency
f_p = 0.24;          % Beginning of passband frequency
weights = [10 1];    % Error weighting in stopband/passband
des = [0 1];         % Desired frequency response in stopband/passband
L = 20;              % L = (M-1)/2 (half of the impulse response length)
K = 50;              % Number of points in the frequency grid

% For the frequency grid, we take K points between 0 and 0.5
freq = linspace(0, 0.5, K);

% Compute the matrices U and W and the vector d
U = ??? % To be completed
d = ??? % To be completed
W = ??? % To be completed

% Set up the optimization problem
A = ??? % To be completed
b = ??? % To be completed

[hh fval] = fmincon(@objfun, ones(L+1, 1), A, b);

% Use the solution of the optimization problem hh to compute the complete
% M-point impulse response.
h = ??? % To be completed
%Plot h and H.
```

Define your objective function as nested function, so that it can access the parent function's variables

```
function f = objfun(hh)

    % Returns the squared error for the given filter coefficients h
    % To be completed...
```