ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 7 Signal Processing for Communications
Homework 4 March 14, 2011, INF 213 - 10:15-12:00

This is the first graded homework. You can discuss this homework with your colleagues but you have to write down your own solution. Please indicate all people with whom you have discussed the homework on the top of the first page. If we find similarities beyond the indicated contacts you will get zero points. Please hand in the homework on March 21st during the exercise session from 10:15 till 12pm. No exceptions.

Problem 1 (This Only Looks Complex).

i) Show that

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}mn} = \begin{cases} N & \text{when } m = kN \text{ , } k \text{ integer} \\ 0 & \text{otherwise} \end{cases}$$

holds.

ii) Find N complex solutions of the equation $x^N - 1 = 0$.

iii) Determine the value of the summation given below in terms of N, when k is an integer such that $0 < k < \lfloor N/2 \rfloor$

$$\sum_{n=0}^{N-1} \cos^2(\frac{2\pi}{N}kn).$$

Problem 2 (Black Box). Let $x[n]$ be a discrete real-valued signal of length N. For $k = 0, \ldots, N-1$, let $F_N^k(\cdots)$ be the functional which maps $x[n]$ into its k-th Fourier coefficient $X[k]$, i.e., $X[k] = F_N^k(x[0], \ldots, x[N-1])$. In a similar manner. Let $G_N^k(\cdots)$ denote this functional assuming that $x[n]$ is complex valued.

i) [Make it Complex] Assume that you have given the functionals $F_N^k(\cdots)$ as black boxes. How can you implement the functionals $G_N^k(\cdots)$ using these black boxes?

ii) [Revert to Invert] Assuming that you have given the functionals $G_N^k(\cdots)$ as black boxes. How can you implement the inverse DFT in terms of these black boxes?

iii) [Even if this Sounds Odd] Assume we are given the functionals $G_N^k(\cdots)$ as black boxes. How can you compute the DFT of a complex-valued signal $x[n]$ of length $2N$ in terms of these black boxes?

Problem 3. Let $\{a_n\}$ denote a family of complex numbers. Recall that we say that $\{a_n\}$ is *summable* if and only if for every $\epsilon > 0$ there exists a finite set $J_\epsilon \in \mathbb{N}$ such that for every finite set $K \in \mathbb{N}$, $\left|\sum_{n \in K} a_n\right| < \epsilon$ if $K \cap J_\epsilon = \emptyset$. Equivalently, as we have shown in class, $\{a_n\}$ is *summable* with sum S if for every ϵ, there exists a finite set $J_\epsilon \in \mathbb{N}$, such that for every finite set $K \in \mathbb{N}$, $\left|S - \sum_{n \in K} a_n\right| < \epsilon$ if $K \supseteq J_\epsilon$.

Also recall that we say that the sum $\sum_{n=1}^{\infty} a_n$ converges *absolutely* if and only if $\sum_{n=1}^{\infty} |a_n| < \infty$.

i) Is the family $\{a_n\}$, with $a_n = \frac{(-1)^n}{n}$, summable?

ii) Does the sum $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge absolutely?

iii) Prove that over $\mathbb{C}$, a family $\{a_n\}$ is summable if and only if the corresponding sum $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Problem 4. In this exercise, we want to approximate a continuous function $p(\cdot) : [0, 1] \to \mathbb{R}$ by a polynomial with degree at most n, i.e.

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

such that the total squared error defined as

$$\text{err} = \int_0^1 |p(x) - p_n(x)|^2 \, \mathrm{d}x$$

is minimum.

Let $C[0, 1]$ be the Hilbert space of functions $f(\cdot) : [0, 1] \to \mathbb{C}$ with the following inner product:

$$\langle f(x), g(x) \rangle = \int_0^1 f(x) g^*(x) \mathrm{d}x.$$

Recall that the norm induced by the inner product is $\| f(x) \| = \sqrt{\langle f(x), f(x) \rangle}$. Consider the set of polynomial functions $\{1, x, x^2, \cdots, x^n\}$ on the interval $[0, 1]$.

i) Prove that the set is linearly independent.

Recall: We say that the set of functions $\{1, x, x^2, \cdots, x^n\}$ is linearly independent if

$$\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n = 0, \quad \forall x \in [0, 1], \qquad (*)$$

implies that all coefficients $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ are zero. Otherwise we say that the set is linearly dependent.

ii) Since the set is linearly independent, it spans an $(n+1)$-dimensional subspace, call it $\mathbb{P}(n)$. This subspace contains all polynomial functions of degree at most n. Explain how to find an orthonormal basis, call it $\{u_0(x), u_1(x), \cdots, u_n(x)\}$, for this subspace.

iii) Let $q(x) = \sum_{i=0}^{n} b_i u_i(x) \in \mathbb{P}(n)$ where $b_i \in \mathbb{C}$ for $i = 0, 1, ..., n$. The total squared error between $q(x)$ and $p(x)$ is $\| p(x) - q(x) \|^2$. Use the projection theorem to find $p_n(x) \in \mathbb{P}(n)$ such that

$$\| p(x) - p_n(x) \| = \min_{q(x) \in \mathbb{P}(n)} \| p(x) - q(x) \| .$$

iv) For $p(x) = \sin(\frac{\pi}{2} x)$, find $p_2(x) = a_0 + a_1 x + a_2 x^2$.