Problem 1 (Multirate Identities).

(i) We can see that after downsampling, we get $X_D(z) = \frac{1}{2}(X(z^{1/2}) + X(-z^{1/2}))$, which we filter with $H(z)$, and we get as output $\frac{H(z)}{2}(X(z^{1/2}) + X(-z^{1/2}))$.

If we first filter by $H(z^2)$, we get $X_{\text{filtered}}(z) = X(z)H(z^2)$, which gets downsampled, and we get as output $\frac{1}{2}(X(z^{1/2})H(z) + X(-z^{1/2})H(z))$.

(ii) Following a similar method as for the first part, it is easy to see that the output in both cases is $X(z^2)H(z^2)$.

Problem 2 (Polyphase Implementation of Downsampling).

Let $z[n] = x[n] * h[n]$, then $y[n] = z[2n]$. Thus,

$$\begin{aligned} y[n] = z[2n] &= \sum h[2n-m]x[m] \\ &= \sum h[2n-2k]x[2k] + \sum h[2n-2k+1]x[2k-1] \\ &= \sum e_0[n-k]x[2k] + \sum e_1[n-k]x[2k-1] \end{aligned}$$

Therefore, the two systems are equivalent.

Problem 3 (Quatization Error).

(i) Since $f_x(x)$ is a probability density function, we have :

$$\int_0^1 f_x(x)dx = 1 \Rightarrow b = \frac{1}{2}$$

(ii) The interval $[0,1]$ divided into 2^r points. Then the points in the interval $\left[\frac{i}{2^r}, \frac{i+1}{2r}\right)$ map to $\frac{i}{2^r}$. Assume that $\hat{x} = \frac{i}{2^r}$, then :

$$\begin{aligned} P(\hat{X} = \hat{x} = \frac{i}{2^r}) &= p\{\frac{i}{2^r} \le x \le \frac{i+1}{2^r}\} \\ &= \int_{\frac{i}{2^r}}^{\frac{i+1}{2^r}} f_x(x)dx = \frac{1}{2^{r+1}} + \frac{1}{4}((\frac{i+1}{2^r})^2 - (\frac{i}{2^r})^2) = \frac{1}{2^{r+1}} + \frac{1}{2^{r+2}} \cdot \frac{2i+1}{2^r} \end{aligned}$$

(iii)

$$\begin{aligned} p_e &= \int_0^1 (x-\hat{x})^2 f_x(x)dx = \sum_{i=0}^{2^r-1} \int_{\frac{i}{2^r}}^{\frac{i+1}{2^r}} (x - \frac{i}{2^r})^2 f_x(x)dx \\ &= \sum_{i=0}^{2^r-1} \int_{\frac{i}{2^r}}^{\frac{i+1}{2^r}} x^2 + \sum_{i=0}^{2^r-1} \int_{\frac{i}{2^r}}^{\frac{i+1}{2^r}} \frac{i^2}{2^{2r}} f_x(x)dx - \sum_{i=0}^{2^r-1} \frac{2i}{2^r} \int_{\frac{i}{2^r}}^{\frac{i+1}{2^r}} x f_x(x)dx \\ &= \int_0^1 x^2 f_x(x)dx + \sum_{i=0}^{2^r-1} \frac{i^2}{2^{2r}}(\frac{1}{2^{r+1}} + \frac{2i+1}{2^{2r+2}}) + \sum_{i=0}^{2^r-1} \frac{2i}{2^r}(\frac{2i+1}{2^{2r+2}} + \frac{1}{6}((\frac{i+1}{2^r})^3 - (\frac{i}{2^r})^3)). \end{aligned}$$

Problem 4 (Minimize Quantization Error)**.**

Let $P(a,t)$ be the power of the quantization error. Then,

$$\begin{aligned}
P(a,t) &= \int (x-\hat{x})^2 f_x(x)dx = \int (x - Q\{x\})^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{-t}^{t} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + 2\int_{t}^{\infty} (x-a)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + 2a^2 \int_{t}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - 4a \int_{t}^{\infty} \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= 1 + 2a^2 \int_{t}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - 4a \int_{t}^{\infty} \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx
\end{aligned}$$

To minimize $P(a,t)$, we take its partial derivatives and put them equal to zero, i.e. :

$$\begin{aligned}
\frac{\partial P(a,t)}{\partial a} &= 4a \int_{t}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx - 4 \int_{t}^{\infty} \frac{xe^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = 0 \\
&\Rightarrow a \int_{t}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \int_{t}^{\infty} \frac{xe^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \qquad (1)
\end{aligned}$$

and

$$\frac{\partial P(a,t)}{\partial t} = -2a^2 e^{-t^2/2} + 4ate^{-t^2/2} = 0 \Rightarrow a = 2t$$

By putting $t = a/2$ in (1), we can find the proper value for a.