Problem 1 (Multirate Identities). Prove the following two identities:

(i) Downsampling by 2 followed by filtering by \( H(z) \) is equivalent to filtering by \( H(z^2) \) followed by downsampling by 2.

(ii) Filtering by \( H(z) \) followed by upsampling by 2 is equivalent to upsampling by 2 followed by filtering by \( H(z^2) \).

Problem 2 (Polyphase Implementation of Downsampling). Consider the downsampling system given in Fig. 1, where \( H(z) \) is an arbitrary filter with impulse response \( h[n] \).

We define

\[ e_0[n] = h[2n], \quad e_1[n] = h[2n + 1]. \]

Prove that the system of Fig. 2 is equivalent to the one given in Fig. 1.

Problem 3 (Quantization Error). Consider the quantizer \( Q \) which takes \( X \) that has a value in the interval \([0, 1]\) and outputs \( \hat{X} \), the first \( r \) bits of its binary expansion. Assume we feed \( X \) with the following probability density function into the quantizer:

\[ f_X(x) = \frac{1}{2} + bx. \]

(i) Find \( b \).

(ii) Compute \( P(\hat{X} = \hat{x}) \).

(iii) Compute the power of the quantization error.
Problem 4 (Minimize Quantization Error). Consider a stationary i.i.d. random process $X[n]$ whose samples have normal distribution $N(0,1)$. The process is quantized with a 3 points quantizer $Q\{\}$ with the following characteristic:

$$Q\{x\} = \begin{cases} 
+ a & x \geq t, \\
0 & -t < x < t, \\
- a & x \leq -t.
\end{cases}$$

Find the proper values for $a$ and $t$ in order for the power of the quantization error to be minimized.