The Entropy of Markov Trajectories

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Measures of information
by Prof. Vignat
Complexity as Thermodynamic Depth

- Physical System with macroscopic states.
- **Depth** as Universal measure of state’s complexity.
- Depth of a state depends on $p_i$, the probability of the trajectory taken (starting at a given state).
- **Average Depth** of a state $d$: $D(d) = - \sum_i p_i \ln p_i$
- Computation of trajectories Entropy needed.
Entropy of Markov Trajectories

• Finite state irreducible Markov chain (MC) $P$
• Stationary distribution $\Pi(j) = \sum \Pi(i) P_{ij} \forall j$
• A trajectory $t_{ij} \in T_{ij}$ from state $i$ to state $j$ is a path with initial state $i$, final state $j$ and no intervening state $j$.
• Trajectory $t_{ij} = ix_1 x_2 \ldots x_k j$ conditional probability

$$p(t_{ij}) = P_{ix_1} P_{x_1 x_2} \ldots P_{x_k j}.$$
Entropy of Markov Trajectories

• Entropy rate irreducible MC

\[ H(X) = - \sum_{ij} \Pi(i) P_{ij} \log P_{ij}. \]

• Irreducibility of the MC implies

\[ \sum_{t_{ij} \in T_{ij}} p(t_{ij}) = 1. \]

• Entropy of the trajectory from \( i \) to \( j \) is

\[ H_{ij} \equiv H(T_{ij}) = - \sum_{t_{ij} \in T_{ij}} p(t_{ij}) \log p(t_{ij}). \]
Fundamental Recurrence

- Let $P_{i.}$ be the $i$th row of $P$
- First step entropy $H(P_{i.}) = - \sum_j P_{ij} \log(P_{ij})$.
- Chain rule for entropy
  $$H_{ij} = H(P_i) + \sum_{k \neq j} P_{ik} H_{kj}$$
- Matrix recurrence to solve
  $$H = H^* + PH - PH\Delta.$$
Fundamental recurrence

\[ H = H^* + PH - PH_\Delta. \]

First Step Entropies

Transition probabilities

\[ \text{diag}(H_{11}, H_{22}, \ldots, H_{nn}) \]
Back to origin $i$

Entropy rate

$$H_{ii} = \frac{H(X)}{\pi(i)}.$$

Stationary Probability for state $i$

Entropy of random trajectory from state $i$ back to state $i$
Closed form expression

If $P$ is the transition matrix of an irreducible finite state Markov chain, then the matrix $H$ of trajectory entropies is given by

$$H = K - \tilde{K} + H_\Delta,$$

where

$$K = (I - P + A)^{-1} (H^* - H_\Delta),$$

$$\tilde{K}_{ij} = K_{jj} \quad \text{for all } i, j,$$

$$A_{ij} = \Pi(j) \quad \text{for all } i, j,$$

$$H^*_{ij} = H(P_i) \quad \text{for all } i, j,$$

and

$$(H_\Delta)_{ij} = \begin{cases} \frac{H(X)}{\pi(i)} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$
Proof structure

1. Aperiodic Markov chains.
2. Periodic Markov chains.
Theorem

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Conclusions

• Depth as a measure of a system complexity.
• Irreducible finite state Markov chain.
• Closed form expression for trajectory entropies.
• What if we condition on an intermediary state?