The Entropy of Markov Trajectories

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Measures of information
by Prof. Vignat

Complexity as Thermodynamic Depth

- Physical System with macroscopic states.
- Depth as Universal measure of state's complexity.
- Depth of a state depends on p_i, the probability of the trajectory taken (starting at a given state).
- Average Depth of a state *d*: $D(d) = -\sum_i p_i \ln p_i$
- Computation of trajectories Entropy needed.

Entropy of Markov Trajectories

- Finite state irreducible Markov chain (MC) P
- Stationary distribution $\Pi(j) = \sum_i \Pi(i) P_{ij} \; \forall j$
- A trajectory $t_{ij} \in T_{ij}$ from state *i* to state *j* is a path with initial state *i*, final state *j* and no intervening state *j*.
- Trajectory $t_{ij} = i x_1 x_2 ... x_k j$ conditional probability

$$p(t_{ij}) = P_{ix_1} P_{x_1 x_2} \dots P_{x_k j}.$$

Entropy of Markov Trajectories

- Entropy rate irreducible MC

$$H(X) = -\sum_{ij} \Pi(i) P_{ij} \log P_{ij}.$$

- Irreducibility of the MC implies

$$\sum_{t_{ij} \in T_{ij}} p(t_{ij}) = 1.$$

- Entropy of the trajectory from *i* to *j* is

$$H_{ij} \equiv H(T_{ij}) = -\sum_{t_{ij} \in T_{ij}} p(t_{ij}) \log p(t_{ij}).$$

Fundamental Recurrence

- Let $P_{i.}$ be the *i* th row of P
- First step entropy $H(P_{i.}) = -\sum_j P_{ij}\log(P_{ij})$.
- Chain rule for entropy

$$H_{ij} = H(P_i) + \sum_{k \neq j} P_{ik} H_{kj}$$

- Matrix recurrence to solve

$$H = H^* + PH - PH_{\Delta}.$$

Fundamental recurrence

First Step
Entropies

Transition
probabilities

$$H = H^* + PH - PH_\Delta.$$

Trajectories
Entropy

$$\mathrm{diag}(H_{11}, H_{22}, \ldots, H_{nn})$$

Back to origin i

Entropy rate

Entropy of random trajectory from state *i* back to state *i*

$$H_{ii} = \frac{H(X)}{\pi(i)}.$$

Stationary Probability for state *i*

Closed form expression

If P is the transition matrix of an irreducible finite state Markov chain, then the matrix H of trajectory entropies is given by

$$H = K - \tilde{K} + H_\Delta,$$

where

$$\begin{aligned} K &= (I - P + A)^{-1} (H^* - H_\Delta), \\ \tilde{K}_{ij} &= K_{jj} \qquad \text{for all } i, j, \\ A_{ij} &= \Pi(j) \qquad \text{for all } i, j, \\ H^*_{ij} &= H(P_i) \qquad \text{for all } i, j, \end{aligned}$$

and

$$(H_\Delta)_{ij} = \begin{cases} \frac{H(X)}{\pi(i)} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof structure

1. Aperiodic Markov chains.
2. Periodic Markov chains.
3. Solution Uniqueness.

Theorem

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$$(H_{\Delta})_{ij} = \begin{cases} \frac{H(X)}{\pi(i)} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Conclusions

- Depth as a measure of a system complexity.
- Irreducible finite state Markov chain.
- Closed form expression for trajectory entropies.
- What if we condition on an intermediary state ?