Problem 1 (DFT Revisit).

(i) Define \( Y[k] = X[k](-1)^k \) and \( y[n] = x[(n - \frac{N}{2}) \mod N] \).

On the other hand, \( X_1[k] = \text{real}\{Y[k]\} = \frac{1}{2}(Y[k] + Y^*[k]) \). Since \( x[n] \) is real and therefore \( y[n] \) is real, \( Z[k] = Y^*[k] = Y[-k \mod N] \) and then, \( z[n] = y[-n \mod N] \).

Thus, \( x_1[n] = y[n] + y[-n \mod N] \) and \( x_2[n] = \frac{1}{2}(y[n] - y[-n \mod N]) \) and since the DFT is linear function, we can say that

\[
X_1[k] = \frac{1}{2}(Y[k] + Y[-k \mod N]),
\]

and

\[
X_2[k] = \frac{1}{2}(Y[k] - Y[-k \mod N]).
\]

Problem 2 (Limits of Z-transform).

(i) \( X(1) = \sum_{n=-\infty}^{\infty} x[n] = (a) \sum_{n=0}^{\infty} x[n] \)

(\( a \) is correct since \( x[n] \) is causal. It shows the limit of the series \( \sum_{n=0}^{\infty} x[n] \). If the ROC of \( X(z) \) contains the unit circle, then it has limit and the limit is equal to \( X(1) \).

(ii) \( \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] \).

(iii) \( \lim_{z \to \infty} z(X(z) - x[0]) = x[1] \). The result follows from the fact that:

\[
\lim_{z \to \infty} z(X(z) - x[0]) = \lim_{z \to \infty} z \left( \sum_{n=0}^{\infty} x[n] z^{-n} - x[0] \right) = \lim_{z \to \infty} \sum_{n=0}^{\infty} x[n] z^{-(n-1)} = \lim_{z \to \infty} \sum_{n=0}^{\infty} x[n+1] z^{-n} = x[1].
\]

(iv) \( X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \Rightarrow \frac{dX(z)}{dz} = \sum_{n=0}^{\infty} -nx[n] z^{-(n+1)} \).

Therefore, \( -z \frac{dX(z)}{dz} = \sum_{n=0}^{\infty} nx[n] z^{-n} \).

It is the z-transform of \( nx[n] \).
(v) from (iv), \(-z\frac{dX(z)}{dz} = \sum_{n=0}^{\infty} nx[n]z^{-n} = \sum_{n=1}^{\infty} nx[n]z^{-n}\).

Hence,

\[
\lim_{z \to \infty} -z^2 \frac{dX(z)}{dz} = \lim_{z \to \infty} \sum_{n=1}^{\infty} nx[n]z^{-(n-1)} = \lim_{z \to \infty} \sum_{n=0}^{\infty} (n+1)x[n+1]z^{-n} = x[1]
\]

**Problem 3 (Stochastic Processes).**

(i)

\[
m_x = E[x[n]] = E[\sin(\omega n + \theta)] = E[\sin(\omega n) \cos(\theta) + \cos(\omega n) \sin(\theta)]
\]

\[
= \sin(\omega n)E[\cos(\theta)] + \cos(\omega n)E[\sin(\theta)].
\]

\[
E[\sin(\theta)] = \int_{-\infty}^{\infty} \sin(\theta)f_\theta(\theta)d\theta = \int_{0}^{2\pi} \sin(\theta)\frac{1}{2\pi}d\theta = \left. \frac{-1}{2\pi} \cos(\theta) \right|_{0}^{2\pi} = 0.
\]

In the same manner, \(E[\cos(\theta)] = 0\).

\[
R_X[\ell, k] = E[X[\ell]X[k]] = E\{\sin(\omega \ell + \theta) \sin(\omega k + \theta)\}.
\]

We know that \(\sin(\varphi_1)\sin(\varphi_2) = \frac{1}{2}(\cos(\varphi_1 - \varphi_2) - \cos(\varphi_1 + \varphi_2))\). Thus,

\[
R_X[\ell, k] = E\left\{\frac{1}{2}\left[\cos(\omega(\ell - k)) - \cos(\omega(\ell + k) + 2\theta)\right]\right\}
\]

\[
= \frac{1}{2}E[\cos(\omega(\ell - k))] - \frac{1}{2}E\{\cos(\omega(\ell + k) + 2\theta)\}
\]

\[
= \frac{1}{2}\cos(\omega(\ell - k)) - \frac{1}{2}E\{\cos(\omega(\ell + k) + 2\theta)\} = \frac{1}{2}\cos(\omega(\ell - k)).
\]

The last equality is due to

\[
E\{\cos(\omega(\ell + k) + 2\theta)\} = E\{\cos(\omega(\ell + k))\cos(2\theta) - \sin(\omega(\ell + k))\sin(2\theta)\}
\]

\[
= \cos(\omega(\ell + k))E\{\cos(2\theta)\} - \sin(\omega(\ell + k))E\{\sin(2\theta)\}.
\]

and \(E\{\cos(2\theta)\} = \int_{0}^{2\pi} \cos(2\theta)\frac{1}{2\pi}d\theta = \left. \frac{1}{4\pi} \sin(2\theta) \right|_{0}^{2\pi} = 0\).

Similarly, \(E\{\sin(2\theta)\} = 0\).

Since \(m_X\) is fixed and \(R_X[\ell, k]\) is only a function of \(\ell - k\), we can say that \(x[n]\) is a wide-sense stationary signal.

(ii) Let’s first compute the impulse response of this filter.

\[
h[n] = \delta[n] + \beta \delta[n - 1]
\]

Therefore,

\[
H(e^{j2\pi f}) = 1 + \beta e^{-j2\pi f}.
\]

On the other hand,

\[
P_X(e^{j2\pi f}) = FT\{R_X[k]\} = \frac{1}{2\pi} \left[ \tilde{\delta}(2\pi f - \omega) - \tilde{\delta}(2\pi f + \omega) \right].
\]
Therefore,
\[ P_Y(e^{j2\pi f}) = |H(e^{j2\pi f})|^2 P_X(e^{j2\pi f}) \]
\[ = |H(e^{j2\pi f})|^2 \frac{1}{2j} \left[ \delta(2\pi f - \omega) - \delta(2\pi f + \omega) \right] \]
\[ = |H(e^{j\omega})|^2 \frac{1}{2j} \left[ \delta(2\pi f - \omega) - \delta(2\pi f + \omega) \right]. \]

(iii) We should compute \( P_X(e^{j2\pi f}) : \)
\[ P_X(e^{j2\pi f}) = \sum_{k=-\infty}^{\infty} R_X[k] e^{-j2\pi ft} = \sigma^2 \sum_{k=-\infty}^{-1} \alpha^{-k} e^{-j2\pi ft} + \sigma^2 \sum_{k=0}^{\infty} \alpha^k e^{-j2\pi ft} \]
\[ = \sigma^2 \left( \frac{\alpha e^{j2\pi f}}{1 - \alpha e^{j2\pi f}} + \frac{1}{1 - \alpha e^{-j2\pi f}} \right) = \sigma^2 \left( \frac{1 - \alpha^2}{1 + \alpha^2 - 2\alpha \cos(2\pi f)} \right) \]

More over,
\[ |H(e^{j2\pi f})|^2 = |1 + \beta e^{-j2\pi f}|^2 = |1 + \beta \cos(2\pi f) - j\beta \sin(2\pi f)|^2 \]
\[ = (1 + \beta^2 + 2\beta \cos(2\pi f)). \]
Thus,
\[ P_Y(e^{j2\pi f}) = (1 + \beta^2 + 2\beta \cos(2\pi f)) \sigma^2 \left( 1 + \alpha^2 - 2\alpha \cos(2\pi f) \right). \]

(iv) \( Y[n] \) corresponds to a white noise, if it power spectral density is a constant value for all frequencies. Therefore,
\[ P_Y(e^{j2\pi f}) = (1 + \beta^2 + 2\beta \cos(2\pi f)) \sigma^2 \left( 1 + \alpha^2 - 2\alpha \cos(2\pi f) \right) = \text{const.} \]

if and only if
\[ \frac{1 + \beta^2 + 2\beta \cos(2\pi f)}{1 + \alpha^2 - 2\alpha \cos(2\pi f)} = \text{const.} \]

Hence, we can conclude that \( \beta = -\alpha. \)

Problem 4 (Min. Mean Squared Error Estimator*).

(i) We should verify the following three properties of inner product :
- Positivity :
\[ \langle u, u \rangle = \int uu^* P_{X,Y}(x, y) dxdy = \int |u|^2 P_{X,Y}(x, y) dxdy \geq 0 \]
- Linearity :
\[ \langle au + bw, v \rangle = E((au + bw)v^*) = aE(uv^*) + bE(wv^*) \]
\[ = a\langle u, v \rangle + b\langle w, v \rangle \]

The above equalities are due to linearity of expectation function.
- Conjugate symmetry:

\[ \langle u, v \rangle = E(uv^*) = (E(u^*v))^* = (\langle v, u \rangle)^* \]

(ii) Since it is unbiased estimator,

\[ E(X) = E(\hat{X}) = E(aY + b) = aE(Y) + b \]  

(1)

Since it is minimum mean squared estimator,

\[ E \left\{ (X - \hat{X})^2 \right\} = E \left\{ X^2 + \hat{X}^2 - 2X\hat{X} \right\} 
  = E(X^2) + E \left\{ \hat{X}^2 - 2X\hat{X} \right\} \]

\[ E(X^2) \] is fixed and we should minimize the second component:

\[ E \left\{ \hat{X}^2 - 2X\hat{X} \right\} = E \left\{ (aY + b)^2 - 2X(aY + b) \right\} 
  = E((a^2Y^2 + b^2 + 2abY) - 2aXY - 2bX) 
  = a^2E(Y^2) + b^2 + 2abE(Y) - 2bE(X) - 2aE(XY) \]

\[-b^2 \text{ from (1)} \]

\[ = a^2E(Y^2) - 2aE(XY) - b^2 \]

We know that \( b = E(X) - aE(Y) = m_X - am_Y \). Thus,

\[ E \left\{ \hat{X}^2 - 2X\hat{X} \right\} = a^2E(Y^2) - 2aE(XY) - (m_X - am_Y)^2 = f(a) \]

To minimize \( E \left\{ \hat{X}^2 - 2X\hat{X} \right\} = f(a) \), we can take the derivative from \( f(a) \) and set it equal to zero,

\[ f'(a) = 2aE(Y^2) - 2E(XY) + 2m_Y(m_X - am_Y) = 0 \]

\[ \Rightarrow a = \frac{E(XY) - m_Xm_Y}{E(Y^2) - m_Y^2}, \quad b = m_X - am_Y \]

(iii) Shortly, the subspace of random variable \( Y \) contains \( Y \) and all continuous functions \( f(y) \). Assume that \( p(y) \) is the minimum mean squared estimator, i.e.

\[ \arg \min_{f(y)} \langle x - f(y), x - f(y) \rangle = p(y) \]

According to projection theorem, since \( x - p(y) \) has the minimum norm for all members of subspace. \( p(y) \) is projection of \( x \) on that subspace and, as we know, it is the projection iff

\[ \langle x - p(y), f(y) \rangle = 0 \]

\( (x - p(y) \) is orthogonal with all the members of subspace)

According to hint 2, \( E(X|Y) = g(Y) \) has such property and, therefore, \( g(Y) = E(X|Y) \) is the projection of \( X \) on the subspace of \( Y \) and it is the best minimum squared error estimator.