Problem 1. Consider the channel $C$ depicted in Figure 1 (a). In this channel $C$, the input $X$ is taken from the set of real numbers and $Z \sim \mathcal{N}(0, 1)$ is an additive random Gaussian noise with distribution $\mathcal{N}(0, 1)$ which is independent from $X$. The observation $Y$ is equal to $X + Z$. Notice that $Y$ is a real valued random variable. We learned that the capacity of this channel with power constraint $P$ is equal to $\frac{1}{2} \log(1 + P)$.

In practice it is not convenient to have to deal with a continuous input alphabet. But, as we will see, if we use a suitable discrete input alphabet (rather than the capacity achieving Gaussian distribution) then we can achieve rates arbitrarily close to the optimal setup.

(i) Consider the following channel $C_2$ in which $X \in \{-1, 1\}$, $Z \sim \mathcal{N}(0, 1)$ is a Gaussian noise and $Y_2$ is defined as $Y_2 = X + Z$. More precisely, we are only allowing binary inputs. Show that the resulting channel is symmetric and so the optimum input distribution is the uniform one. There is no closed-form solution for the capacity of this channel. But you can write it down as an integral. State this integral and use Matlab (or Mathematica) to plot the capacity as a function of $P$. Compare this to the capacity of the channel $C$. Any comments?

(ii) So far we have quantized the input. Sometimes it is also convenient to quantize the output. What is the capacity of the channel $C'_2$ defined as following:

$$Y'_2 = \begin{cases} 1 & \text{if } Y_2 > 0, \\ -1 & \text{if } Y_2 \leq 0 \end{cases}$$

Plot this capacity also on the previous plot and compare.

Problem 2. Let $Y = X + Z$, where $X$ and $Z$ are independent, zero mean random variables with variances $a$ and $b$, respectively. In this problem we want to prove the following inequality: $h(X|Y) \leq \frac{1}{2} \log(\frac{2\pi eab}{a+b})$.

(i) Prove that $h(X|Y) \leq \frac{1}{2} \log(2\pi eE_Y(\text{Var}(X|Y)))$.

(ii) Show that $E_Y(\text{Var}(X|Y)) \leq \frac{ab}{a+b}$.

(iii) Conclude that $h(X|Y) \leq \frac{1}{2} \log(\frac{2\pi eab}{a+b})$.

Hint: For part (ii) you can use the following facts

- $E_X((E(X|Y) - X)^2) = \text{Var}(X|Y)$. (Consider both sides as functions of $Y$.)

- In this problem, the minimum of $E((X - \hat{X})^2)$ over all estimator for $X$ in terms of the observation $Y$ (i.e the minimum over all the functions of $Y$) is obtained when $\hat{X}(y) = E(X|Y = y)$. 

**Problem 3.** Consider the channel with input $X$ and output $(Y_1, Y_2)$ where $Y_1 = X + Z_1$ and $Y_2 = X + Z_2$ in which $Z_1$ and $Z_2$ are two Gaussian additive random noises both independent from $X$. Suppose that $(Z_1, Z_2) \sim \mathcal{N}(0, K)$ where $K$ is the covariance matrix of $Z_1$ and $Z_2$. Let $P$ be the power constrain of the input. (See Figure 2) Compute the capacity of the channel if

$$K = \begin{bmatrix} M & M\sigma \\ M\sigma & M \end{bmatrix}$$

and

(i) $\sigma = 1$

(ii) $\sigma = \frac{1}{2}$

(iii) $\sigma = -1$

**Problem 4.** Consider a vector Gaussian channel described as follows:

$$Y_1 = X + Z_1$$
$$Y_2 = Z_2$$

in which $X \in \mathbb{R}$ is the input to the channel constrained in power to $P$; $Z_1$ and $Z_2$ are jointly Gaussian random variables with $E(Z_1) = E(Z_2) = 0, E(Z_1^2) = \sigma_1^2, E(Z_2^2) = \sigma_2^2$ and $E(Z_1Z_2) = \sigma_3^2$, with $\sigma_3 \leq \sigma_1, \sigma_2$, and independent of the channel input.

(i) What is the capacity of the channel 1 depicted in Figure 4, (a)?

(ii) What is the capacity of the channel 2 depicted in Figure 4 (b)?

(iii) What is the capacity of the channel 3 depicted in Figure 4 (c)?
Figure 3: Problem 4, (a): Channel 1, (b): Channel 2, (c): Channel 3.