Handout 10 Advanced Digital Communications

Homework 4 Solutions November 1, 2010

PROBLEM 1. (a)

$$\begin{aligned}
\Pr(R \le r) &= \Pr(|z| \le r) && (1)\\
&= \int\int P_z(x+jy)\,dx\,dy && (2)\\
&= \int_0^r \int_0^{2\pi} P_z(u\cos\theta + ju\sin\theta)u\,du\,d\theta && (3)\\
&= \int_0^r \int_0^{2\pi} P_z(ue^{j\theta})u\,du\,d\theta && (4)\\
& && (5)
\end{aligned}$$

Differentiating with respect to r will yield:

$$P_R(r) = r\int_0^{2\pi} P_z(re^{j\theta})\,d\theta$$

(b)

$$\Pr(U \le u) = \Pr(R^2 \le u) = \Pr(R \le \sqrt{u})$$

Differentiating with respect to u:

$$P_U(u) = \frac{1}{2\sqrt{u}}P_R(\sqrt{u}) = \frac{1}{2}\int_0^{2\pi} P_z(\sqrt{u}e^{j\theta})\,d\theta$$

(c) Since z is circularly symmetric, $P_z(\sqrt{u}e^{j\theta})$ does not depend on θ.

$$P_U(u) = \pi P_Z(\sqrt{u})$$

(d) If x and y are independent and with common density p, we have:

$$P_z(x+jy) = P_z(\sqrt{x^2+y^2}(\cos\phi + j\sin\phi)) = p(x)p(y)$$

Using part (c), we have

$$\begin{aligned}
P_U(x^2+y^2) &= \pi P_z(\sqrt{x^2+y^2}) && (6)\\
&= \pi p(x)p(y) && (7)
\end{aligned}$$

(e) Evaluating $P_U(x^2+y^2)$ at $x=0$ and $y=0$, we would have

$$P_U(x^2+y^2) = \frac{\pi P_u(x^2)P_u(x^2)}{(\pi p(0))^2}$$

Let us define $f(y) = \frac{P_U(u)}{(\pi p(0))^2}$. This is a continous function and satifies $f(a+b) = f(a)f(b)$ for all nonnegative a and b. Using hint we have $f(a) = e^{\beta a}$. Solving for β by integrating $P_U(u)$ and making it equal to 1.

$$\beta = -\pi p^2(0)$$

(f) Combining above we have

$$P_Z(z) = P_Z(|z|) = \frac{1}{\pi} P_U(|z|^2) = \frac{1}{\pi} e^{-\frac{|z|^2}{\sigma^2}}$$

So Z is a Gaussian random variable.

PROBLEM 2. (a)

$$f_{V|U}(\mathbf{v}|a) = \frac{1}{(\pi N_0)^n} e^{\frac{-||\mathbf{v}-a||^2}{N_0}}$$

and

$$f_{V|U}(\mathbf{v}|-a) = \frac{1}{(\pi N_0)^n} e^{\frac{-||\mathbf{v}+a||^2}{N_0}}$$

(b)

$$\text{LLR}(\mathbf{v}) = \log \frac{f_{V|U}(\mathbf{v}|-a)}{f_{V|U}(\mathbf{v}|a)} = \frac{-||\mathbf{v}-\mathbf{a}||^2 + ||\mathbf{v}+\mathbf{a}||^2}{N_0}.$$

(c) ML rule is comparing LLR to the constant zero and because LLR depends on difference of distance of channel output to vectors a and $-a$, ML is a minimum distance detector.

(d)

$$||\mathbf{v}-\mathbf{a}||^2 = ||\mathbf{v}||^2 - \langle \mathbf{v}, \mathbf{a} \rangle - \langle \mathbf{a}, \mathbf{v} \rangle + ||\mathbf{a}||^2$$

and

$$||\mathbf{v}+\mathbf{a}||^2 = ||\mathbf{v}||^2 + \langle \mathbf{v}, \mathbf{a} \rangle + \langle \mathbf{a}, \mathbf{v} \rangle + ||\mathbf{a}||^2$$

As $\langle \mathbf{v}, \mathbf{a} \rangle + \langle \mathbf{a}, \mathbf{v} \rangle = 2\text{Re}\langle \mathbf{v}, \mathbf{a} \rangle$, by substituting in (c) one gets the result.

(e) In the detection, only the real part of the $\langle \mathbf{v}, \mathbf{a} \rangle$ matters and it is important if it is positive or negative, but $|\mathbf{v}, \mathbf{a}|$ preserves none of the above.

(f) No, if v is in this space cv can be out of the space.

PROBLEM 3. (a) We know Fourier transform of sinc function is a rect. So the Fourier transform of sinc^2 is rect $*$ rect. The Fourier transform of

$$\text{sinc}^2(Wt)$$

is

$$\frac{1}{W} \Lambda(\frac{f}{W}),$$

where $\Lambda(f)$ is a triangle function zero valued at $|f| = 1$ and unit valued at $f = 0$.

(b)

$$v(t) = \sum_k u(KT)\text{sinc}(\frac{t}{T} - k) * \text{sinc}^2(Wt) = \frac{1}{2W} \sum_k u(KT) g(t - KT)$$

(c) Because both $u(KT)$ and $g(t - KT)$ are non negative then $v(t)$ is non negative.

(d) In formula $u(t) = \sum_k u(KT)\text{sinc}(\frac{t}{T} - k)$, we set $u(t) = 1$. The result follows.

(e) We take Fourier transform of

$$\sum_k g(\frac{t}{T}-k) = g(\frac{t}{T}) * \sum_k \delta(t-kT)$$

We get

$$G(fT)\sum_k \delta(f-kT) = \frac{1}{T}\frac{1}{W}\delta(f) = 2\delta(f)$$

so $\sum_k g(\frac{t}{T}-k) = 2$

(f)

$$v(t) = \sum_k u(kT)g(t-kT) \leq \sum_k g(t-kT) = 2$$

(g)

$$\begin{aligned} |v(t)| &= |\sum_k u(kT)g(t-kT)| && (8)\\ &\leq \sum_k |u(kT)||g(t-kT)| && (9)\\ &\leq \sum_k |g(t-kT)| && (10)\\ &= \sum_k g(t-kT) && (11)\\ &= 2 && (12) \end{aligned}$$

PROBLEM 4. (a) For $0 \leq t \leq 1$, we have

$$p(t)p(t-1) = p(t) - p^2(t) = 0$$

So the inner product $\langle p(t), p(t-1)\rangle$ which is an integration is also zero.

(b) For $|k| > 2$, $p(t)p(t-k) = 0$. The case $|k| = 1$ follows from part (a).

(c) For $|k| = 1$, $p(t)p(t-1)e^{j2\pi mt} = (p(t) - p^2(t))e^{j2\pi mt} = 0$. For the other values of k the product of two functions is zero everywhere.

(d)

$$\begin{aligned} \langle p(t), p(t)e^{j2\pi mt}\rangle &= \int_{-1}^{1} p^2(t)e^{j2\pi mt}\,dt && (13)\\ &= \int_{-1}^{1} p(t)e^{j2\pi mt}\,dt && (14)\\ &= \int_{-1}^{0} p(t)e^{j2\pi mt}\,dt + \int_{0}^{1} p(t)e^{j2\pi mt}\,dt && (15)\\ &= \int_{0}^{1} p(t-1)e^{j2\pi m(t-1)}\,dt + \int_{0}^{1} p(t)e^{j2\pi mt}\,dt && (16)\\ &= \int_{0}^{1} p(t-1)e^{j2\pi mt}\,dt + \int_{0}^{1} p(t)e^{j2\pi mt}\,dt && (17)\\ &= \int_{0}^{1} (p(t-1)+p(t))e^{j2\pi mt}\,dt && (18)\\ &= \int_{0}^{1} e^{j2\pi mt}\,dt = 0 && (19) \end{aligned}$$

(e) Yes, this property will hold: $p(t)$ being orthogonal to $p(t-k)e^{j2\pi mt}$ is equivalent to $\hat{p}(f)$ being orthogonal to $\hat{p}(f-m)e^{-j2\pi kt}$ for every nonzero m and k.