Problem 1.

(a) For the channel of problem 1, homework 8, show that the canonical factorization is

\[ Q(D) + \frac{q_0}{\text{SNR}_{\text{MFB}}} = \gamma_0(1 - r_2 D^{-1})(1 - r_2^* D). \]

What is \( \gamma_0 \) in terms of \( a \) and \( b \)? Please do not do this from scratch. You have done much of the work for this in problem 2, HW7.

(b) Find \( A(D) \) and \( W(D) \) for the MMSE DFE.

(c) Give an expression for \( \gamma_{\text{MMSE-DFE}} \). Compute its values for \( a = 0, .5, 1 \) for the \( E_x = 1 \) and \( \sigma^2 = 0.1 \) Sketch \( \gamma_{\text{MMSE-DFE}} \) as in problem 1, HW8. Compare with your sketches from problem 1, HW8.

Hint.

\[ \gamma_{\text{MMSE-DFE}} = 10 \log_{10} \left( \frac{\text{SNR}_{\text{MFB}}}{\text{SNR}_{\text{MMSE-DFE}}} \right) \]

Problem 2.

Consider the following discrete time channel:

\[ y_k = x_k + x_{k-1} + z_k \]

where \( \{z_k\} \) is i.i.d. gaussian with variance \( \sigma^2 \) and is independent of \( \{x_k\} \). Moreover suppose that \( \{x_k\} \), are independently and uniformly chosen from \( \{A, -A\} \).

(a) Find the optimal estimator \( \hat{x}_k(y_k) \), an estimator that uses only the current channel output. Compute its error probability. Hint: Consider the cases \( x_{k-1} = x_k \) and \( x_{k-1} \neq x_k \) separately.

(b) Find the optimal estimator \( \hat{x}_k^{\text{DFE}}(y_k, \hat{x}_{k-1}) \) that uses the previously estimated symbol to detect the last sent symbol, then:

(i) Find \( \Pr(\hat{x}_k \neq x_k) \) conditioned on \( \hat{x}_{k-1} = x_{k-1} \)

(ii) Find \( \Pr(\hat{x}_k \neq x_k) \) conditioned on \( \hat{x}_{k-1} \neq x_{k-1} \)

(iii) Find \( \Pr(\hat{x}_k \neq x_k) \) and compare it with the probability of error of the previous estimator.

Problem 3. Consider the scalar discrete-time inter symbol interference channel,

\[ y_k = \sum_{n=0}^{\nu} p_n x_{k-n} + z_k, \quad k = 0, \ldots, N - 1, \tag{1} \]

where \( z_k \sim \mathcal{CN}(0, \sigma_z^2) \) and is i.i.d., independent of \( \{x_k\} \). Let us employ a cyclic prefix as done in OFDM, i.e.,

\[ x_{-l} = x_{N-1-l}, \quad l = 0, \ldots, \nu. \]
As done in class given the cyclic prefix,

\[
y = \begin{bmatrix} y_{N-1} \\ \vdots \\ y_0 \end{bmatrix} = \begin{bmatrix} p_0 & \ldots & p_\nu & 0 & \ldots & 0 & 0 \\ 0 & p_0 & \ldots & p_{\nu-1} & p_\nu & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & p_0 & \ldots & \ldots & p_\nu \\ p_\nu & 0 & \ldots & 0 & 0 & p_0 & \ldots & p_{\nu-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ p_1 & \ldots & p_\nu & 0 & \ldots & 0 & 0 & p_0 \end{bmatrix} \begin{bmatrix} x_{N-1} \\ \vdots \\ x_0 \end{bmatrix} + \begin{bmatrix} z_{N-1} \\ \vdots \\ z_0 \end{bmatrix}.
\]  

(2)

In the derivation of OFDM we used the property that

\[
P = V^*DV,
\]

(3)

where

\[
V_{p,q} = \frac{1}{\sqrt{N}} \exp\left(-j\frac{2\pi}{N}(p-1)(q-1)\right)
\]

and \(D\) is the diagonal matrix with

\[
D_{l,l} = d_l = \sum_{n=0}^{\nu} p_n e^{-j\frac{2\pi}{N}nl}.
\]

Using this we obtained

\[
Y = V y = DX + Z,
\]

where \(X = Vx, Z = Vz\). This yields the parallel channel result

\[
Y_l = d_l X_l + Z_l.
\]

(4)

If the carrier synchronization is not accurate, then (1) gets modified as

\[
y(k) = \sum_{n=0}^{\nu} e^{j2\pi f_0 k} p_n x_{k-n} + z_k, \quad k = 0, \ldots, N-1
\]

(5)

where \(f_0\) is the carrier frequency offset. If we still use the cyclic prefix for transmission, then (2) gets modified as

\[
\begin{bmatrix} y(N-1) \\ \vdots \\ y(0) \end{bmatrix} = \begin{bmatrix} p_0 e^{j2\pi f_0(N-1)} & \ldots & p_\nu e^{j2\pi f_0(N-1)} & 0 & \ldots & 0 & 0 \\ 0 & \ldots & e^{j2\pi f_0 p_0} & \ldots & e^{j2\pi f_0 p_\nu} \\ e^{j2\pi f_0 p_1} & \ldots & e^{j2\pi f_0 p_\nu} & 0 & \ldots & 0 & e^{j2\pi f_0 p_0} \end{bmatrix} \begin{bmatrix} x_{N-1} \\ \vdots \\ x_0 \end{bmatrix} + \begin{bmatrix} z_{N-1} \\ \vdots \\ z_0 \end{bmatrix}.
\]

(6)

i.e.,

\[y = Hx + z\]

Note that

\[H = SP,\]

where \(S\) is a diagonal matrix with \(S_{l,l} = e^{j2\pi f_0(N-l)}\) and \(P\) is defined as in (2).
(a) Show that for $Y = V y$, $X = V x$,

$$Y = G X + Z \tag{7}$$

and prove that

$$G = V S V^* D.$$

(b) If $f_0 \neq 0$, we see from part (a) that $G$ is no longer a diagonal matrix and therefore we do not obtain the parallel channel result of (4). We get inter-carrier interference (ICI), i.e., we have

$$Y_l = G_{l,l} X_l + \sum_{q \neq l} G(l, q) X_q + Z_l, \quad l = 0, \ldots, N - 1,$$

which shows that the other carriers interfere with $X_l$. Compute the SINR (signal-to-interference plus noise ratio). Assume $\{X_l\}$ are i.i.d, with $\mathbb{E}|X_l|^2 = \mathcal{E}_x$. You can compute the SINR for the particular $l$ and leave the expression in terms of $\{G(l, q)\}$.

(c) Find the filter $W_l$, such that the MMSE criterion is fulfilled,

$$\min_{W_l} \mathbb{E}|W_l^* Y - X_l|^2.$$

You can again assume that $\{X_l\}$ are i.i.d with $\mathbb{E}|X_l|^2 = \mathcal{E}_x$ and that the receiver knows $G$. You can now state the answer in terms of $G$.

(d) Find an expression for $G_{l,q}$ in terms of $f_0, N, \{d_l\}$. Using Taylor series show that $G_{l,q} \approx \theta c_{l,q}$ for small $\theta$. What can you say about the ICI if $\theta \ll 1/N^2$?

*Hint:* Use the summation of the geometric series

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}.$$