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School of Computer and Communication Sciences

Handout 15 Advanced Digital Communications
Homework 7 November 22, 2010

PROBLEM 1. Suppose we have a linear time invariant channel, i.e.,

$$Y(D) = Q(D)X(D) + Z(D);$$

with $Q(D) = Q^*(D^{-*})$. Also there is another process $U(D) = H(D)X(D)$; which we want to estimate.

(a) Given observations $\{y_k\}$, find the linear estimator

$$\hat{U}(D) = W(D)Y(D)$$

which minimizes the mean-squared error, i.e.,

$$W(D) = \text{argmin}_{W(D)} E||u_k - \hat{u}_k||^2$$

You can assume that $\{X_k\}$ and $\{Z_k\}$ are independent and that

$$S_x(D) = E_x$$

and

$$S_z(D) = N_0 Q(D).$$

(b) Given the optimum linear MMSE estimator given in part (a) we define the error as

$$e_k = u_k - \hat{u}_k$$

Find the power spectral density of $\{e_k\}$, $S_E(D)$.

(c) If $H(D) = 1$, can you comment on the operation performed in part (a)?

PROBLEM 2. Consider estimating the real zero-mean scalar x from:

$$\mathbf{y} = \mathbf{h}x + \mathbf{w}$$

where $\mathbf{w} \sim \mathcal{N}(0, \frac{N_0}{2}\mathbf{I})$ is uncorrelated with x and $\mathbf{h}$ is a fixed vector in $\mathcal{R}^n$.

(a) Consider the scaled linear estimate $\mathbf{c}^t\mathbf{y}$ (with the normalization $||\mathbf{c}|| = 1$):

$$\hat{x} = a\mathbf{c}^t\mathbf{y} = (a\mathbf{c}^t\mathbf{h})x + a\mathbf{c}^t\mathbf{z} \tag{1}$$

Show that the constant a that minimizes the mean square error $(x - \hat{x})^2$ is equal to

$$\frac{\mathbb{E}[x^2]|\mathbf{c}^t\mathbf{h}|}{\mathbb{E}[x^2]|\mathbf{c}^t\mathbf{h}|^2 + \frac{N_0}{2}} \tag{2}$$

(b) Calculate the minimal mean square error (denoted by MMSE) of the linear estimate in (1) (by using the value of a in (2)). Show that

$$\frac{\mathbb{E}[x^2]}{\text{MMSE}} = 1 + \text{SNR} = 1 + \frac{\mathbb{E}[x^2]|\mathbf{c}^t\mathbf{h}|^2}{\frac{N_0}{2}} \tag{3}$$

For every fixed linear estimator $\mathbf{c}$, this shows the relationship between the corresponding SNR and MMSE (of an appropriately scaled estimate).

(c) In particular, relation (3) holds when we optimize over all $\mathbf{c}$ leading to the best linear estimator. Find the value of vector $\mathbf{c}$ (with the normalization $||\mathbf{c}|| = 1$) by minimizing the MMSE derived in part (b). Compute optimal MMSE.

Hint. Use Cauchy-Schwarz inequality.

Problem 3. (Linear Estimation) Consider the additive noise model given below,

$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_2$$

Let $X, Y_1, Y_2, Z_1, Z_2 \in \mathcal{C}$, i.e. they are complex random variables. Moreover, assume X, Z_1 and Z_2 are zero mean and Z_1 and Z_2 are independent of X.

(a) Assume the following: $\mathbb{E}[|X|^2] = \mathcal{E}_x, \mathbb{E}[|Z_1|^2] = \mathbb{E}[|Z_2|^2] = 1$ and $\mathbb{E}[Z_1 Z_2^*] = 0$. Given Y_1, Y_2 find the best minimum mean squared error linear estimator $\hat{X}$, where the optimization criterion is $\mathbb{E}[|X - \hat{X}|^2]$.

(b) If $\mathbb{E}[Z_1 Z_2^*] = \frac{1}{\sqrt{2}}$, what is the best MMSE linear estimator of X ?

(c) If $\mathbb{E}[Z_1 Z_2^*] = 1$, what is the best MMSE linear estimator of X ?

Problem 4. Let Y_a and Y_b be two separate observations of a zero mean random variable X such that

$$\begin{aligned} Y_a &= H_a X + V_a \\ \text{and } Y_b &= H_b X + V_b, \end{aligned}$$

where $\{V_a, V_b, X\}$ are mutually independent and zero-mean random variables, and $V_a, V_b, X, Y_a, Y_b \in \mathcal{C}$.

(a) Let $\widehat{X}_a$ and $\widehat{X}_b$ denote the linear MMSE estimators for X given Y_a and Y_b respectively. That is

$$\begin{aligned} W_a &= \text{argmin}_{W_a} \mathbb{E}[||X - W_a Y_a||^2], \\ W_b &= \text{argmin}_{W_b} \mathbb{E}[||X - W_b Y_b||^2] \end{aligned}$$

and

$$\widehat{X}_a = W_a Y_a \quad \text{and} \quad \widehat{X}_b = W_b Y_b.$$

Find $\widehat{X}_a$ and $\widehat{X}_b$ given that

$$\mathbb{E}[XX^*] = \sigma_x^2,\ \mathbb{E}[V_a V_a^*] = \sigma_a^2,\ \mathbb{E}[V_b V_b^*] = \sigma_b^2.$$

Also, find the error variances,

$$\begin{aligned} P_a &= \mathbb{E}[(X - \widehat{X}_a)(X - \widehat{X}_a)^*] \\ P_b &= \mathbb{E}[(X - \widehat{X}_b)(X - \widehat{X}_b)^*] \end{aligned}$$

(b) We have the following identities,

$$\begin{aligned}
\mathbf{R}_x\mathbf{H}^*\left[\mathbf{H}\mathbf{R}_x\mathbf{H}^*+\mathbf{R}_v\right]^{-1} &= \left[\mathbf{R}_x^{-1}+\mathbf{H}^*\mathbf{R}_v^{-1}\mathbf{H}\right]^{-1}\mathbf{H}^*\mathbf{R}_v^{-1} \\
\mathbf{R}_x-\mathbf{R}_x\mathbf{H}^*\left[\mathbf{H}\mathbf{R}_x\mathbf{H}^*+\mathbf{R}_v\right]^{-1}\mathbf{H}\mathbf{R}_x &= \left[\mathbf{R}_x^{-1}+\mathbf{H}^*\mathbf{R}_v^{-1}\mathbf{H}\right]^{-1}
\end{aligned}$$

where

$$\mathbf{H}=\begin{bmatrix}H_a\\ H_b\end{bmatrix}, \mathbf{R_v}=\begin{bmatrix}\sigma_a^2 & 0\\ 0 & \sigma_b^2\end{bmatrix}, \mathbf{R_x}=\sigma_x^2.$$

Prove that

$$P_a^{-1}\widehat{X}_a=\frac{H_a^*}{\sigma_a^2}Y_a,\quad P_b^{-1}\widehat{X}_b=\frac{H_b^*}{\sigma_b^2}Y_b. \tag{4}$$

and

$$P_a^{-1}=\frac{1}{\sigma_x^2}+\frac{H_aH_a^*}{\sigma_a^2},\quad P_b^{-1}=\frac{1}{\sigma_x^2}+\frac{H_bH_b^*}{\sigma_b^2}. \tag{5}$$

(c) Now we find the estimator $\widehat{X}$, given both observations Y_a and Y_b, i.e.,

$$\begin{pmatrix}Y_a\\ Y_b\end{pmatrix}=\begin{pmatrix}H_a\\ H_b\end{pmatrix}X+\begin{pmatrix}V_a\\ V_b\end{pmatrix}.$$

We want to find the linear MMSE estimate

$$\widehat{X}=\begin{pmatrix}U_a & U_b\end{pmatrix}\begin{pmatrix}Y_a\\ Y_b\end{pmatrix},$$

where

$$\begin{pmatrix}U_a & U_b\end{pmatrix}=\mathrm{argmin}_{(U_a,U_b)}\mathbb{E}[||X-\widehat{X}||^2]$$

and define the corresponding error variance

$$P=\mathbb{E}[(X-\widehat{X})(X-\widehat{X})^*].$$

Use (4), (5) to show that

$$\begin{aligned}
P^{-1}\widehat{X} &= P_a^{-1}\widehat{X}_a+P_b^{-1}\widehat{X}_b \\
\text{and}\quad P^{-1} &= P_a^{-1}+P_b^{-1}-\frac{1}{\sigma_x^2}.
\end{aligned}$$