Problem 1. Consider the transmit pulse:

\[ g_{TX}(t) = \text{sinc} \left( \frac{t}{T} \right) \cdot \text{sinc} \left( \frac{t}{2T} \right) \]

(a) Prove that it satisfies the Nyquist condition at symbol rate \( \frac{1}{T} \).

(b) If \( g_{TX}(t) \) is used for Nyquist signaling using 8-PSK (the constellation formed by 8 points equally distributed on the unit circle) at 6 Mbit/s, what is the minimum required channel bandwidth?

(c) For the setting in (b), suppose that the complex baseband channel has impulse response \( g_{C}(t) = \delta(t - 0.5T) - \frac{1}{2} \delta(t - 1.5T) + \frac{1}{4} \delta(t - 2.5T) \). What is the minimum number of states in the trellis for MLSE using the Viterbi algorithm?

Problem 2. Consider the noisy ISI channel given by

\[ Y_i = X_i + X_{i-1} + Z_i \]

where \( X_i \) and \( Y_i \) are the channel input and output, respectively at time index \( i \), \( Z \) is a sequence of i.i.d. Gaussian random variables, with zero mean and unit variance and \( x_i \in \{-1, 1\} \).

Calculate the symbol-wise MAP estimate of \( X \), using the BCJR algorithm, if the received sequence \( Y = [0.28, -0.54, -0.46, 2.26, 1.52] \). You may assume that the channel is in state +1 at the beginning and at the end of the sequence. Compare this to the decoding estimate from the MLSE decoder.

Problem 3. Consider the following real channel,

\[ y = hx + z, \]

where \( x \in \mathbb{R} \) is a random variable, with \( E[x] = 0 \) and \( E[x^2] = \mathcal{E} \), \( h \) is a fixed real vector, and \( z \) is a zero-mean random vector with covariance matrix \( I \), chosen independently of the value of \( x \).

(i) An estimator \( \hat{x} = \mathcal{F}(y) \) is said to be unbiased if \( E[\hat{x}|x] = x \).

(1) Consider the mentioned channel, what is the constraint for a linear estimator, i.e., \( \hat{x} = a^t y \) to be unbiased?

(2) Find the unbiased linear estimator that minimizes the mean squared error:

\[ \sigma^2_{\text{unbiased}} = E[(x - \hat{x})^2] \]

and the value of \( \sigma^2_{\text{unbiased}} \) for this estimator.

Hint. By the Cauchy-Schwartz inequality, \( (a^t a)(h^t h) \geq |a^t h|^2 \).
(ii) In this part, we don’t restrict ourselves to unbiased estimators. Suppose $F$ is a linear estimator. Find the linear estimator that minimizes the mean squared error $\sigma^2 = E[(x - \hat{x})^2]$ and the value of $\sigma^2$ for this estimator. 

Hint. Consider $\hat{x} = a^T y$, and minimize $\sigma^2$ with respect to vector $a$. Do the minimization in two steps, first assume $a^T h = c$, and minimize $\sigma^2$ with respect to vector $a$ with the constraint $a^T h = c$, and then minimize the result with respect to $c$.

(iii) Compare the two “signal to noise ratio”s $\mathcal{E}/\sigma^2_{\text{unbiased}}$ and $\mathcal{E}/\sigma^2$.

(iv) Now assume $x$ is equally likely to be $+1$ or $-1$. Suppose a decision is made by quantizing the estimate $\hat{x}$ from either part (i) or (ii) to $\pm 1$. Which estimator would you choose to minimize the probability of error?