Problem 1. Suppose $Z$ is a complex random variable with density $p_Z$.

(a) Let $R = |Z|$. Show that the density $p_R$ of $R$ is given by

$$p_R(r) = r \int_0^{2\pi} p_Z(r \exp(j\theta)) \, d\theta.$$  

Hint: Write $\Pr(R \leq r)$ as an integral over $x$ and $y$, then use polar coordinates.

(b) Let $U = R^2$. Show that its density is given by

$$p_U(u) = \frac{1}{2} \int_0^{2\pi} p_Z(\sqrt{u} \exp(i\theta)) \, d\theta.$$  

(c) Suppose now that $Z$ is circularly symmetric. Show that

$$p_U(u) = \pi p_Z(\sqrt{u}).$$  

(d) Again suppose $Z$ is circularly symmetric. Let $X$ and $Y$ be its real imaginary parts. We know that $X$ and $Y$ are identically distributed, call the common density $p$. Suppose that $X$ and $Y$ are independent. Show that

$$p_U(x^2 + y^2) = \pi p(x)p(y).$$  

(e) Under the assumptions of (d), conclude that

$$p_U(x^2 + y^2) = \frac{1}{\pi p(0)^2} p_U(x^2)p_U(y^2).$$

Assuming that $p_U$ is continuous show that it must be given by

$$p_U(u) = \alpha \exp(-\alpha u), \quad u \geq 0,$$

where $\alpha = \pi p(0)^2$. Hint: The only continuous functions $f$ that satisfies $f(a + b) = f(a)f(b)$ are those for which $f(a) = \exp(\beta a)$ for some $\beta$.

(f) Show that if $Z$ is circularly symmetric complex random variable with independent real and imaginary parts, then $Z$ must be Gaussian.

Problem 2. Let $Z = (Z_1, \ldots, Z_n)^T$ be a vector of complex iid Gaussian rvs with iid real and imaginary parts, each $N(0, N_0)$). The input $U$ is binary antipodal, taking on values $a$ or $-a$, where $a = (a_1, \ldots, a_n)^T$ is an arbitrary complex $n$-vector. The observation $V$ is $U + Z$, and the probability density of $Z$ is given by

$$f_Z(z) = \frac{1}{(\pi N_0)^n} e^{\sum_{j=1}^n -|z_j|^2 / N_0} = \frac{1}{(\pi N_0)^n} e^{-||z||^2 / N_0}.$$
(a) Give expressions for \( f_{V|U}(v|a) \) and \( f_{V|U}(v|-a) \).

(b) Show that the log likelihood ratio for the observation \( v \) is given by
\[
\text{LLR}(v) = -||v - a||^2 + ||v + a||^2.
\]

(c) Explain why this implies that ML detection is minimum distance detection (defining the distance between two complex vectors as the norm of their difference).

(d) Show that \( \text{LLR}(v) \) can also be written as \( \frac{4\text{Re}((v,a))}{N_0} \).

(e) The appearance of the real part, \( \text{Re}((v,a)) \), in part (d) is surprising. Point out why log likelihood ratios must be real. Also explain why replacing \( \text{Re}((v,a)) \) by \( |(v,a)| \) in the above expression would give a non-sensical result in the ML test.

(f) Does the set of points \( \{v : \text{LLR}(v) = 0\} \) form a complex vector space?

**Problem 3.** (Amplitude-limited functions) Sometimes it is important to generate baseband waveforms with bounded amplitude. This problem explores pulse shapes that can accomplish this.

(a) Find the Fourier transform of \( g(t) = \text{sinc}^2(Wt) \). Show that \( g(t) \) is bandlimited to \( f \leq W \) and sketch both \( g(t) \) and \( \hat{g}(f) \). [Hint. Recall that multiplication in the time domain corresponds to convolution in the frequency domain.]

(b) Let \( u(t) \) be a continuous real \( L_2 \) function baseband-limited to \( f \leq W \) (i.e. a function such that \( u(t) = \sum_k u(kT) \text{sinc}(\frac{f}{W}) - k) \), where \( T = \frac{1}{2W} \). Let \( v(t) = u(t) * g(t) \). Express \( v(t) \) in terms of the samples \( \{u(kT); k \in \mathbb{Z}\} \) of \( u(t) \) and the shifts \( \{g(t - kT); k \in \mathbb{Z}\} \) of \( g(t) \). [Hint. Use your sketches in part (a) to evaluate \( g(t) * \text{sinc}(\frac{f}{W}) \).]

(c) Show that if the \( T \)-spaced samples of \( u(t) \) are nonnegative, then \( v(t) \geq 0 \) for all \( t \).

(d) Explain why \( \sum_k \text{sinc}(\frac{f}{W} - k) = 1 \) for all \( t \).

(e) Using (d), show that \( \sum_k g(\frac{k}{W} - c) = c \) for all \( t \) and find the constant \( c \). [Hint. Use the hint in (b) again.]

(f) Now assume that \( u(t) \), as defined in part (b), also satisfies \( u(kT) \leq 1 \) for all \( k \in \mathbb{Z} \). Show that \( v(t) \leq c \) for all \( t \).

(g) Allow \( u(t) \) to be complex now, with \( |u(kT)| \leq 1 \). Show that \( v(t) \leq c \) for all \( t \).

**Problem 4.** (Orthogonal sets) The function \( \text{rect}(\frac{t}{T}) \) has the very special property that it, plus its time and frequency shifts, by \( kT \) and \( \frac{f}{T} \), respectively, form an orthogonal set. The function \( \text{sinc}(\frac{t}{T}) \) has the same property. We explore other functions that are generalizations of \( \text{rect}(\frac{t}{T}) \) and which, as you will show in parts (a)-(d), have this same interesting property. For simplicity, choose \( T = 1 \). These functions take only the values 0 and 1 and are allowed to be nonzero only over \([-1; 1]\) rather than \([-\frac{1}{2}; \frac{1}{2}]\) as with \( \text{rect}(\frac{t}{T}) \). Explicitly, the functions considered here satisfy the following constraints:
\[
\begin{align*}
p(t) &= \quad p^2(t) \quad \text{for all } t \ (0/1 \text{ property}); \\
p(t) &= \quad 0 \quad \text{for } |t| > 1; \\
p(t) &= \quad p(-t) \quad \text{for all } t \ (\text{symmetry}); \\
p(t) &= \quad 1 - p(t - 1) \quad \text{for } 0 \leq t \leq 1/2.
\end{align*}
\]
Note: because of property (3), condition (4) also holds for $1/2 < t \leq 1$. Note also that $p(t)$ at the single points $t = \pm \frac{1}{2}$ does not affect any orthogonality properties, so you are free to ignore these points in your arguments.

(a) Show that $p(t)$ is orthogonal to $p(t - 1)$.

*Hint.* Evaluate $p(t)p(t - 1)$ for each $t \in [0; 1]$ other than $t = \frac{1}{2}$.

(b) Show that $p(t)$ is orthogonal to $p(t - k)$ for all integer $k \neq 0$.

(c) Show that $p(t)$ is orthogonal to $p(t - k)e^{j2\pi mt}$ for integer $k \neq 0$ and $m \neq 0$.

(d) Show that $p(t)$ is orthogonal to $p(t)e^{j2\pi mt}$ for integer $m \neq 0$.

*Hint.* Evaluate $p(t)e^{j2\pi mt} + p(t - 1)e^{j2\pi m(t-1)}$.

(e) Let $h(t) = \hat{p}(t)$ where $\hat{p}(f)$ is the Fourier transform of $p(t)$. If $p(t)$ satisfies properties (1)-(4), does it follow that $h(t)$ has the property that it is orthogonal to $h(t - k)e^{j2\pi mt}$ whenever either the integer $k$ or $m$ is nonzero?

*Note:* almost no calculation is required in this problem.