ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 7 Advanced Digital Communications

Homework 3 October 18, 2010

Problem 1.

(a) By Markov bound, for any positive s, we have

$$\Pr(Z \geq b) = \Pr(e^{sZ} \geq e^{sb}) \leq E\big(e^{s(Z-b)}\big), \qquad s \geq 0.$$

In the following parts of the exercise we assume $x > 0$ because otherwise it is easy to see the bounds do not hold.

(b)

$$\begin{aligned} Q(x) &= \Pr(z \geq x) && (1)\\ &\leq \frac{E\big(e^{sZ}\big)}{e^{sx}} && (2)\\ &= \frac{e^{\frac{s^2}{2}}}{e^{sx}} && (3)\\ &\leq e^{-\frac{x^2}{2}} && (4)\end{aligned}$$

In the third step we use the fact that the integral of the pdf of a gaussian random variable is 1.

(c)

$$\begin{aligned} Q(x) &= \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt && (5)\\ &= \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{t}{t}e^{-\frac{t^2}{2}}\,dt && (6)\\ &= -\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{t^2}{2}}}{t}|_x^\infty - \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{1}{t^2}e^{-\frac{t^2}{2}}\,dt && (7)\\ &\leq \frac{1}{\sqrt{2\pi x^2}}e^{-\frac{x^2}{2}} && (8)\\ & && (9)\end{aligned}$$

For upperbound we have

$$\begin{aligned} Q(x) &= \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt && (10)\\ &= \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{t}{t}e^{-\frac{t^2}{2}}\,dt && (11)\\ &= -\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{t^2}{2}}}{t}|_x^\infty - \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{t}{t^3}e^{-\frac{t^2}{2}}\,dt && (12)\\ &= -\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{t^2}{2}}}{t}|_x^\infty + \frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{t^2}{2}}}{t^3}|_x^\infty + \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{3}{t^4}e^{-\frac{t^2}{2}}\,dt && (13)\\ &\geq (1 - \frac{1}{x^2})\frac{1}{\sqrt{2\pi x^2}}e^{-\frac{x^2}{2}}. && (14)\end{aligned}$$

(d) Let $t = y + x$, and we have

$$Q(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_0^\infty e^{\frac{-y^2}{2} - xy}\, dy$$

It is obvious that

$$e^{\frac{-y^2}{2}} \leq 1$$

By mean value theorem and taylor expansion for some positive value y_*, we have

$$e^{\frac{-y^2}{2}} = 1 - \frac{y^2}{2} + \frac{y_*^4}{8} \geq 1 - \frac{y^2}{2}$$

We know

$$\int_0^\infty e^{-xy}\, dy = \frac{1}{x}$$

and

$$\int_0^\infty \frac{y^2}{2} e^{-xy}\, dy = \frac{1}{x^3}$$

Putting these facts together will give the bounds.

(e) We have

$$\Pr(|x_1| \leq x, |x_2| \leq x) = \Pr(|x_1| \leq x) \Pr(|x_2| \leq x) = (1 - 2Q(x))^2$$

(f) We have

$$\Pr(|x_1|^2 + |x_2|^2 \leq x) = \int_0^{2\pi} \int_0^x \frac{1}{2\pi} e^{-\frac{r^2}{2}}\, r\, dr\, d\theta = 1 - e^{-\frac{x^2}{2}}.$$

(g) Circle is contained in the square and we ignore $Q^2(x)$. We have the result.

Problem 2.

(a) As the constellation has M points in $N = M$ dimensions is spectral efficiency is $\log_2(M)/N = \log_2(M)/M$. The energy per bit is $E_b = \mathcal{E}/\log_2 M$.

(b) The distance between signals i and j with $i \neq j$ is

$$\|a_i - a_j\|^2 = \|a_i\|^2 + \|a_j\|^2 - 2\langle a_i, a_j\rangle = 2\mathcal{E} - 2\mathcal{E}\delta_{ij} = 2\mathcal{E}$$

So the distance between any two points is $\sqrt{2\mathcal{E}}$. Thus, $d_{\min}^2 = 2\mathcal{E}$, and since for any constellation point i all the other $(M-1)$ points are at this distance, each point has $(M-1)$ nearest neighbors.

(c) If signal i is sent, an error will be made if the received point is closer to some other point j. Thus,

$$\Pr(\text{Error}|i) = \Pr(\cup_{j\neq i} E_{ij}|i) \leq \sum_{j\neq i} \Pr(E_{ij}|i)$$

where E_{ij} is the event that the received point lies closer to j than i. Since

$$\Pr(E_{ij}|i) = Q(d_{ij}/(2\sigma)) = Q(\sqrt{\mathcal{E}/(2\sigma^2)})$$

we find that

$$\Pr(\text{Error}) \leq (M-1) Q\Big(\sqrt{\frac{\mathcal{E}}{2\sigma^2}}\Big)$$

(d) Writing $\mathcal{E} = (\log_2 M)E_b$, and using $Q(x) \le (2\pi x^2)^{-1/2}\exp(-x^2/2)$, we find

$$\begin{aligned}
\Pr(\text{Error}) &\le MQ(\sqrt{(E_b/2\sigma^2)\log_2 M}) \\
&\le M\exp\Big(-\frac{E_b}{4\sigma^2}\log_2 M\Big)/\sqrt{\pi(E_b/\sigma^2)\log_2 M} \\
&\le \exp\Big(-\frac{E_b}{4\sigma^2}\log_2 M + \ln M\Big)/\sqrt{\pi(E_b/\sigma^2)\log_2 M} \\
&\le \exp\Big(-\Big[\frac{E_b}{4\sigma^2} - \ln 2\Big]\log_2 M\Big)/\sqrt{\pi(E_b/\sigma^2)\log_2 M}
\end{aligned}$$

Observe now that if $E_b/\sigma^2 > 4\ln 2$, the term in square brackets is positive and as M gets large the right hand side goes to zero exponentially fast in $\log M$.

Note that this result is shows that for reliable communication (i.e., to make Pr(Error) as small as we wish), it is not necessary to use larger and large amounts of enery per bit. As long as the amount of energy we use is larger than a fixed threshold (in our derivation $4\sigma^2\ln 2$) the error probability can be made arbitrarily small. With a more careful derivation we can improve this threshold to $2\sigma^2\ln 2$, in fact this turns out to be best possible.

The spectral efficiency in the limit of large M is $(\log_2 M)/M$ which approaches zero.

e)

$$\begin{aligned}
\Pr\left(\langle a_i, Y\rangle \le T | X = a_i\right) &= \Pr\left(\mathcal{E} + \langle a_i, Z\rangle \le T\right) \\
&= \Pr\big(N(0, \sigma^2\mathcal{E}) \ge \mathcal{E} - T\big) \\
&= Q\left(\frac{\mathcal{E} - T}{\sigma\cdot\sqrt{\mathcal{E}}}\right)
\end{aligned}$$

$$\begin{aligned}
\Pr\left(\langle a_i, Y\rangle \ge T | X = a_j\right) &= \Pr\left(\langle a_i, Z\rangle \ge T\right) \\
&= \Pr\big(N(0, \sigma^2\mathcal{E}) \ge T\big) \\
&= Q\left(\frac{T}{\sigma\cdot\sqrt{\mathcal{E}}}\right)
\end{aligned}$$

f)

$$\begin{aligned}
\Pr(E) &= \sum_{a_j \in A} \Pr(E|X = a_j)\Pr(X = a_j) \\
&= \Pr(E|X = a_1) \\
&\le \Pr\left(\langle a_1, Y\rangle \le T | X = a_1\right) + \sum_{i \ne 1}\Pr\left(\langle a_i, Y\rangle \ge T | X \ne a_i\right) \\
&= Q\left(\frac{\mathcal{E} - T}{\sigma\cdot\sqrt{\mathcal{E}}}\right) + (M-1)Q\left(\frac{T}{\sigma\cdot\sqrt{\mathcal{E}}}\right)
\end{aligned}$$

g)

$$\begin{aligned}
\lim_{M\to\infty} \Pr(E) &\le \lim_{M\to\infty} Q\left(\frac{\mathcal{E}-\alpha\mathcal{E}}{\sigma\cdot\sqrt{\mathcal{E}}}\right) + (M-1)Q\left(\frac{\alpha\mathcal{E}}{\sigma\cdot\sqrt{\mathcal{E}}}\right) \\
&= \lim_{M\to\infty} \exp\left(-\frac{(1-\alpha)^2 E_b \log_2(M)}{2\sigma^2}\right) + (M-1)\exp\Big(-\frac{\alpha^2 E_b \log_2(M)}{2\sigma^2}\Big) \\
&\le \lim_{M\to\infty} M\exp\Big(-\frac{\alpha^2 E_b \log_2(M)}{2\sigma^2}\Big) \\
&= \lim_{M\to\infty} \exp\Big(-\frac{\alpha^2 E_b \log_2(M)}{2\sigma^2} + \ln(M)\Big) \\
&= \lim_{M\to\infty} \exp\Big(-\frac{\alpha^2 E_b \ln(M)/\ln(2) - 2\sigma^2\ln(M)}{2\sigma^2}\Big) \\
&= \lim_{M\to\infty} \exp\Big(-\frac{(\alpha^2 E_b/\ln(2) - 2\sigma^2)\ln(M)}{2\sigma^2}\Big)
\end{aligned}$$

The result of the limit tends to zero iff $(\alpha^2 E_b/\ln(2) - 2\sigma^2) > 0$ therefore by taking $\alpha \to 1$ we obtain that $\Pr(E) \to 0$ iff $E_b/\sigma^2 > 2\ln(2)$.

Problem 3.

1.

$$\begin{aligned}
x(t) &= a\cos\left(2\pi\left(f_c+\frac{1}{T}\right)t\right) + b\cos\left(2\pi\left(f_c+\frac{2}{T}\right)t\right) \\
&= \left(a\cos\left(\frac{2\pi}{T}t\right) + b\cos\left(\frac{4\pi}{T}t\right)\right)\cos(2\pi f_c t) \\
&\quad - \left(a\sin\left(\frac{2\pi}{T}t\right) + b\sin\left(\frac{4\pi}{T}t\right)\right)\sin(2\pi f_c t) \\
&= x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
x_{bb}(t) &= x_I(t) + jx_Q(t) \\
&= a\cos\left(\frac{2\pi}{T}t\right) + ja\sin\left(\frac{2\pi}{T}t\right) + b\cos\left(\frac{4\pi}{T}t\right) + jb\sin\left(\frac{4\pi}{T}t\right) \\
&= a\exp\left(j\frac{2\pi}{T}t\right) + b\exp\left(j\frac{4\pi}{T}t\right)
\end{aligned}$$

Where $\left\{\exp\left(j\frac{2\pi}{T}t\right), \exp\left(j\frac{4\pi}{T}t\right)\right\}$ form an orthonormal basis.

2.

$$\bar{x}_{bb} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\varphi_1(t) = \exp\left(j\frac{2\pi}{T}t\right)$$

$$\varphi_2(t) = \exp\left(j\frac{4\pi}{T}t\right)$$

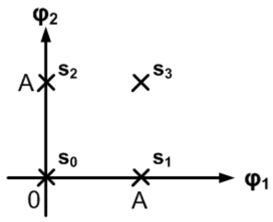


Figure 1: Signal Constellation for $\bar{x}_{bb}$

3.

$$\begin{aligned} E_{bb} = & \frac{1}{4}\sum_{i=0}^{3} E_{s_i} = \frac{1}{4}\sum_{i=0}^{3} \|s_i\|^2 \\ = & \frac{1}{4}(0 + A^2 + A^2 + 2 \cdot A^2) = A^2 \end{aligned}$$

No, this is not a minimum energy constellation. We get the minimum energy constellation by shifting the signal set by $-\frac{1}{4}\sum_{i=0}^{3} \overrightarrow{s_i} = -\frac{1}{2}\begin{pmatrix} A \\ A \end{pmatrix}$; and the origin will be the center of this constellation.

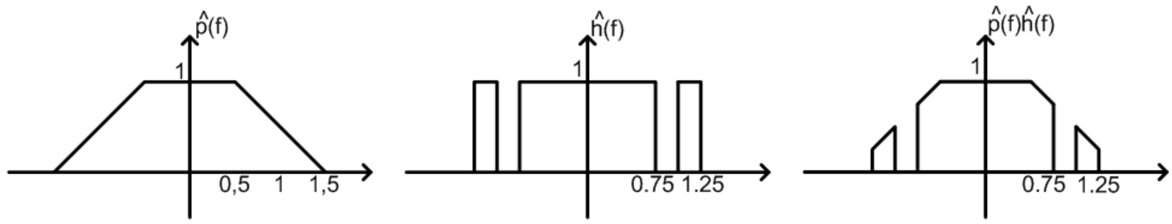
Problem 4.

a)

$$g(kT) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

(Nyquist criterion) We will get a perfect reconstruction if $\hat{g}(f)$ satisfies:

$$\sum_{k=-\infty}^{\infty} \left| \hat{g}\left(f - \frac{k}{T}\right) \right| = T$$



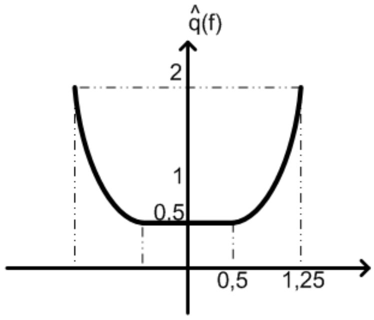
b) Yes, it is possible to find $\hat{q}(f)$ for which there is no intersymbol interference. When,

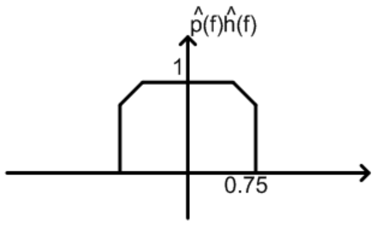
$$\hat{q}(f) = \begin{cases} \frac{1}{2} & |f| \leq 0.5 \\ \frac{1}{2(1.5-f)} & 0.5 < |f| \leq 0.75 \text{ and } 1 < |f| \leq 1.25 \end{cases}$$

$\hat{g}(f)$ satisfies the Nyquist criterion, i.e.

$$\sum_{k=-\infty}^{\infty} |\hat{g}(f - 2k)| = T = \frac{1}{2}$$

and the solution for $\hat{q}(f)$ is non-unique for the intervals $0.75 < |f| \leq 1$ and $|f| > 1.25$. A possible $\hat{q}(f)$ which satisfies the above criteria is as below:





c) This time we have:

and there is no possible solution for $\hat{q}(f)$ so that $\hat{g}(f)$ satisfies the Nyquist criterion; i.e. $\sum_{k=-\infty}^{\infty} |\hat{g}\,(f-2k)| = \frac{1}{2}$ cannot be achieved for any $\hat{q}(f)$ chosen.

d) Intersymbol interference can be avoided by proper choice of $\hat{q}(f)$ iff

$$\sum_{k=-\infty}^{\infty} \hat{p}(f-2k)\hat{h}(f-2k) \neq 0 \quad ,\forall f$$