ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 2 Advanced Digital Communications

Homework 1 September 24, 2010

PROBLEM 1. Let V and W be discrete random variables (rvs) defined with a joint pmf $p_{VW}(v, w)$.

(a) Prove that $E[V + W] = E[V] + E[W]$. Do *not* assume independence.

(b) Prove that if V and W are independent rvs, then $E[V \cdot W] = E[V] \cdot E[W]$.

(c) Find an example where $E[V \cdot W] \neq E[V] \cdot E[W]$ and another example of non-independent V,W where $E[V \cdot W] = E[V] \cdot E[W]$.

(d) Assume that V and W are independent and let σ_V^2 and σ_W^2 be the variances of V and W, respectively. Show that the variance of $V + W$ is given by $\sigma_{V+W}^2 = \sigma_V^2 + \sigma_W^2$.

PROBLEM 2.

(a) For a non-negative integer-valued rv N, show that

$$E[N] = \sum_{n>0} \Pr(N \geq n).$$

(b) Show, with whatever mathematical care you feel comfortable with, that for an arbitrary non-negative rv X,

$$E[X] = \int_0^\infty \Pr(X \geq a)\, da.$$

(c) Derive the Markov inequality, which says that for any $a > 0$ and any non-negative rv X,

$$\Pr(X \geq a) \leq \frac{E[X]}{a}.$$

Hint. Sketch $\Pr(X > a)$ as a function of a and compare the area of the rectangle with horizontal length a and vertical length $\Pr(X \geq a)$ in your sketch with the area corresponding to $E[X]$.

(d) Derive the Chebyshev inequality, which says that

$$\Pr\big(|Y - E[Y]| \geq b\big) \leq \frac{\sigma_Y^2}{b^2}$$

for any rv Y with finite mean $E[Y]$ and finite variance σ_Y^2.

Hint. Use part (c) with $(Y - E[Y])^2 = X$.

Problem 3. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent identically distributed (iid) rvs with the common probability density function $f_X(x)$. Note that $\Pr(X_n = \alpha) = 0$ for all α and that $\Pr(X_n = X_m) = 0$ for $m \neq n$.

(a) Find $\Pr(X_1 \leq X_2)$. (Give a numerical answer, not an expression; no computation is required and a one- or two-line explanation should be adequate.)

(b) Find $\Pr(X_1 \leq X_2; X_1 \leq X_3)$; in other words, find the probability that X_1 is the smallest of $\{X_1, X_2, X_3\}$. (Again, think — do not compute.)

(c) Let the rv N be the index of the first rv in the sequence to be less than X_1; i.e.,

$$\{N = n\} = \{X_1 \leq X_2; X_1 \leq X_3; \ldots; X_1 \leq X_{n-1}; X_1 > X_n\}$$

Find $\Pr(N \geq n)$ as a function of n.

(d) Show that $E[N] = \infty$.

(e) Now assume that $X_1, X_2, \ldots$ is a sequence of iid rvs each drawn from a finite set of values. Explain why you can not find $\Pr(X_1 \leq X_2)$ without knowing the pmf. Explain why $E[N] = \infty$.

Problem 4. Let $X_1, X_2, \ldots, X_n$ be a sequence of n binary iid rvs. Assume that $\Pr(X_m = 0) = \Pr(X_m = 1) = \frac{1}{2}$. Let Z be the parity check on $X_1, X_2, \ldots, X_n$; i.e., $Z = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ (where $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$).

(a) Is Z independent of X_1? (Assume $n > 1$.)

(b) Are $Z, X_1, \ldots, X_{n-1}$ independent?

(c) Are $Z, X_1, \ldots, X_n$ independent?

(d) Is Z independent of X_1 if $\Pr(X_i = 1) \neq \frac{1}{2}$? (You may take $n = 2$ here.)

Problem 5. Consider the binary hypothesis testing problem with MAP decision. Assume that priors are given by $(\pi_0, 1 - \pi_0)$.

(1) Let $V(\pi_0)$ be the overall probability of error. Write the expression for $V(\pi_0)$.

(2) Show that $V(\pi_0)$ is a concave function of π_0 i.e.

$$V(\lambda\pi_0 + (1 - \lambda)\pi_0') \geq \lambda V(\pi_0) + (1 - \lambda)V(\pi_0')$$

for priors $(\pi_0, 1 - \pi_0)$ and $(\pi_0', 1 - \pi_0')$.

Problem 6. Consider Gaussian hypothesis testing with arbitrary priors. Prove that in this case, if y_1 and y_2 are elements of the decision region associated to hypothesis i then so is $\alpha y_1 + (1 - \alpha)y_2$, where $\alpha \in [0, 1]$.