Solutions 9

1. a) One checks that $\mathbb{E}(B_t^2 - t|\mathcal{F}_s^B) = B_s^2 - s$:

$$\mathbb{E}(B_t^2 - B_s^2|\mathcal{F}_s^B) = \mathbb{E}((B_t - B_s)^2 + 2(B_t - B_s)B_s|\mathcal{F}_s^B) = \mathbb{E}((B_t - B_s)^2) + 2\mathbb{E}(B_t - B_s)B_s = t - s.$$

b) One checks that $\mathbb{E}(\exp(B_t - \frac{t}{2})|\mathcal{F}_s^B) = \exp(B_s - \frac{s}{2})$:

$$\mathbb{E}(\exp(B_t - B_s)|\mathcal{F}_s^B) = \mathbb{E}(\exp(B_t - B_s)) = \exp\left(\frac{t-s}{2}\right),$$ since $B_t - B_s \sim \mathcal{N}(0, t-s)$ (see Ex. 4, Hw. 2).

2. (M_t) is clearly adapted to $(\mathcal{F}_t)$ and

(i) $\mathbb{E}(|M_t|) = \mathbb{E}(|\mathbb{E}(X|\mathcal{F}_t)|) \leq \mathbb{E}(\mathbb{E}(|X||\mathcal{F}_t)) = \mathbb{E}(|X|) < \infty$,

(ii) $\mathbb{E}(M_t|\mathcal{F}_s) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}(X|\mathcal{F}_s) = M_s$,

so (M_t) is a martingale with respect to $(\mathcal{F}_t)$.

If moreover X is $\mathcal{F}_T$-measurable for a given $T \in \mathbb{R}_+$, then (M_t) is constant for $t \geq T$.

3. a) For $t > s \geq 0$, we have $\mathbb{E}((M_t - M_s)\, M_s) = \mathbb{E}(\mathbb{E}(M_t - M_s|\mathcal{F}_s)\, M_s) = \mathbb{E}(0\, M_s) = 0$.

b) For $t > s \geq 0$, we have $\mathrm{Cov}(M_t, M_s) = \mathbb{E}(M_t M_s) - \mathbb{E}(M_t)\mathbb{E}(M_s) = \mathbb{E}(M_s^2) - \mathbb{E}(M_0)^2$ is only function of s, therefore of $t \wedge s$.

c) In order to determine c and d, we use part (ii) of the definition of conditional expectation with $g(y) = 1$ and $g(y) = y$, respectively:

$\mathbb{E}(X1) = \mathbb{E}((cY + d)\, 1) = c\,\mathbb{E}(Y) + d$, therefore $d = 0$ (since X and Y are centered).

$\mathbb{E}(XY) = \mathbb{E}((cY)\, Y) = c\,\mathbb{E}(Y^2)$, so $c = \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}$.

d) (X_t) is by definition adapted to its natural filtration and

(i) For $t \geq 0$, $\mathbb{E}(|X_t|) < \infty$, since $\mathbb{E}(X_t^2) < \infty$.

(ii) For $t > s \geq 0$: $\mathbb{E}(X_t|\mathcal{F}_s^X) = \mathbb{E}(X_t|X_s) = \frac{\mathbb{E}(X_t X_s)}{\mathbb{E}(X_s^2)}\, X_s = X_s$ (where we have used the Markov property, part c) and the assumption).