Solutions 7

1. a) X is not a martingale ($\mathbb{E}(\xi_{n+1}|\mathcal{F}_n) = \mathbb{E}(\xi_{n+1}) = 0 \neq \xi_n$), but it is a Markov process.

b) X is a Markov process (as $X_{n+1} = f(X_n, \xi_{n+1})$), but it is not a martingale (except in the case where $a = 1$; notice however that it is neither a submartingale when $a > 1$, nor is it a supermartingale when $a < 1$, as X can take positive and negative values).

c) X is neither a Markov process nor a martingale ($\mathbb{E}(X_{n+1}|\mathcal{F}_n) = \mathbb{E}(\xi_{n+1}|\mathcal{F}_n) + \mathbb{E}(\xi_n|\mathcal{F}_n) = \xi_n$ and $\mathbb{E}(X_{n+1}|X_n) = \mathbb{E}(\xi_{n+1}|X_n) + \mathbb{E}(\xi_n|X_n) = \mathbb{E}(\xi_n|X_n)$, so it does not hold that $\mathbb{E}(g(X_{n+1})|\mathcal{F}_n) = \mathbb{E}(g(X_{n+1})|X_n)$, nor does it hold that $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$).

d) $X_{n+1} = \max(X_n, \xi_{n+1})$, so X is a Markov process, but it is not a martingale (actually, X is an increasing process, i.e. a very particular type of submartingale!).

e) $X_n = \sum_{i=1}^n H_i\, \xi_i$ where $H_1 = 1$, $H_{n+1} = \xi_1 + \ldots + \xi_n$ is predictable and $|H_n| \leq n$ for all n, so X is a martingale. But it is not a Markov process: $X_{n+1} = X_n + (\xi_1 + \ldots + \xi_n)\, \xi_{n+1}$, so the increment of the process between time n and time $n+1$ depends on the whole history of the process.

2. a) By assumption, $\mathbb{P}(\{X \in A, Y \in B\}) = \mathbb{P}(\{X \in A\})\, \mathbb{P}(\{Y \in B\})$

$$= \left(\int_A \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx\right) \cdot \left(\int_B \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy\right)$$
$$= \iint_{A\times B} \frac{1}{\sqrt{(2\pi)^2}} \exp\left(-\frac{x^2+y^2}{2}\right) dxdy.$$

(X, Y) is therefore a centered Gaussian random vector with covariance matrix $K = Id$. This implies that $X + Y$ is a centered Gaussian random variable with variance 2.

b) Z is Gaussian (to show this, use the fact that the distribution of X is symmetric: $\mathbb{P}(\{X \in A\}) = \mathbb{P}(\{-X \in A\})$), but $X + Z$ is not Gaussian, since

$$\mathbb{P}(\{X + Z = 0\}) = \mathbb{P}(\{\mathcal{E} = -1\}) = \frac{1}{2} > 0,$$

which is impossible for a continuous random variable, therefore even more impossible for a Gaussian random variable!

c) $\mathrm{Cov}(X, Z) = \mathbb{E}(XZ) - \mathbb{E}(X)\mathbb{E}(Z) = \mathbb{E}(\mathcal{E}X^2) - 0 = \mathbb{E}(\mathcal{E})\mathbb{E}(X^2) = 0$. Nevertheless,

$$\mathbb{P}(\{X > 1, Z > 1\}) = \mathbb{P}(\{X > 1, \mathcal{E} = +1\}) = \mathbb{P}(\{X > 1\})\, \mathbb{P}(\{\mathcal{E} = +1\}),$$

but $\mathbb{P}(\{\mathcal{E} = +1\}) = \frac{1}{2} \neq \mathbb{P}(\{Z > 1\})$, so X and Z are not independent.

3. The covariance matrix K of X is given by

$$K = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}.$$

It is symmetric for all $a \in \mathbb{R}$, and positive semi-definite if and only if all its eigenvalues are non-negative. Notice that λ is an eigenvalue of K if and only if $\lambda + a - 1$ is an eigenvalue of

$$A = \begin{pmatrix} a & a & a \\ a & a & a \\ a & a & a \end{pmatrix},$$

and that the eigenvalues of A are $3a$, 0, 0. Indeed,

$$0 = \det(A - \lambda I) = (a - \lambda)^3 + 2a^3 - 3(a - \lambda)\, a^2 = (3a - \lambda)\, \lambda^2.$$

Another way to see this is to notice that A has only one non-zero eigenvalue $3a$ corresponding to the eigenvector $(1, 1, 1)$.

The eigenvalues of K are therefore $1 + 2a$, $1 - a$, $1 - a$. They are all non-negative if and only if $a \in [-\frac{1}{2}, 1]$. The answer to the question asked in the problem set is therefore: no.

4. a) X is degenerate if and only if $\text{rank}(K) < 3$ if and only if $\det K = 0$ if and only if one of the eigenvalues of K is zero, that is, $a = 1$ or $a = -\frac{1}{2}$.

If $a = 1$, then $X \sim (U, U, U)$, where $U \sim \mathcal{N}(0, 1)$.

If $a = -\frac{1}{2}$, then $X \sim (U, -\frac{1}{2}U + \frac{\sqrt{3}}{2}V, -\frac{1}{2}U - \frac{\sqrt{3}}{2}V)$ where $U, V \sim \mathcal{N}(0, 1)$ are independent.

Notice that the number of independent random variables needed to build the vector X is equal to the number of non-zero eigenvalues (=rank) of K.

b) Y is degenerate if and only if

$$0 = \det(K) = 1 + 2\, b^4 - b^4 - 2b^2 = (1 - b^2)^2, \quad \text{i.e. if and only if } b = \pm 1.$$

If $b = 1$, then $Y \sim (U, U, U)$, where $U \sim \mathcal{N}(0, 1)$.

If $b = -1$, then $Y \sim (U, -U, U)$, where $U \sim \mathcal{N}(0, 1)$.